Nesting Negations in FO^2 over Finite Words

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Abstract. We consider the fragment Σ_m^2 of two-variable first-order logic $FO^2[<]$ over finite words which is defined by restricting the nesting depth of negations to at most m. Our first result is a combinatorial characterization of Σ_m^2 in terms of so-called rankers. This generalizes a result by Weis and Immerman which we recover as an immediate consequence. Our second result is an effective algebraic characterization of Σ_m^2, *i.e.*, for every integer m one can decide whether a given regular language is definable by a two-variable first-order formula with negation nesting depth at most m. More precisely, for every m we give omega-terms U_m and V_m such that an FO^2-definable language is in Σ_m^2 if and only if its ordered syntactic monoid satisfies the identity $U_m \leqslant V_m$. The proof of this equivalence relies on the ranker characterization.

1 Introduction

A famous result by Büchi, Elgot, and Trakhtenbrot states that a language is definable in monadic second-order logic if and only if it is regular [2, 6, 29]. The algebraic counterpart of regular languages are finite monoids, and in many cases they are the key ingredient for solving decidability problems in this area. For example, by combining a result of Schützenberger [21] with a result of McNaughton and Papert [17], a language is definable in first-order logic FO if and only if its syntactic monoid is finite and aperiodic. The syntactic monoid of a regular language L is the unique minimal monoid recognizing L. It is effectively computable from any reasonable representation of L. One can thus decide definability in FO of a given regular language by checking whether or not its syntactic monoid is aperiodic.

Kamp showed that linear temporal logic is expressively complete for first-order logic over words [9]. Since every modality in linear temporal logic can be defined using three variables, first-order logic with only three different names for the variables (denoted by FO^3) defines the same languages as full first-order logic. This result is often stated as $FO = FO^3$. Using (and reusing) only two variables defines a proper

*Supported by the German Research Foundation (DFG) under grant DI 435/5-1.

FO-Logic	Temporal Logic	Combinatorics	Decidable Criterion	
$\mathrm{FO}^2[<]$	$\mathrm{TL}[\mathsf{XF}, \mathsf{YP}]$	rankers	**DA**	[7, 22, 27]
$\mathrm{FO}^2_{m,n}[<]$	$\mathrm{TL}_{m,n}[\mathsf{XF}, \mathsf{YP}]$	$\equiv^{\mathrm{WI}}_{m,n}$-classes	finitely many languages	[7, 31]
$\mathrm{FO}^2_m[<]$	$\mathrm{TL}_m[\mathsf{XF}, \mathsf{YP}]$		Mal'cev products, identities	[7, 11, 15]
$\Sigma^2_{m,n}[<]$	$\mathrm{TL}^+_{m,n}[\mathsf{XF}, \mathsf{YP}]$	$\leqslant^{\mathrm{WI}}_{m,n}$-ideals	finitely many languages	Thm. 1
$\Sigma^2_m[<]$	$\mathrm{TL}^+_m[\mathsf{XF}, \mathsf{YP}]$		identities $U_m \leqslant V_m$	Thm. 2

subclass. Thérien and Wilke [27] showed that a language is definable in two-variable first-order logic FO^2 if and only if its syntactic monoid belongs to the variety **DA** and, since the latter is decidable, one can effectively check whether a given regular language is FO^2-definable. For further information on the numerous characterizations of FO^2 we refer to [4, 25]. Besides the number of variables, there are two other natural restrictions of first-order logic. The first restriction is the quantifier depth (*i.e.*, the number of nested quantifiers) and the second restriction is the alternation depth (*i.e.*, the number alternations between existential and universal quantification). With respect to decidability questions of the above kind, quantifier depth is not very interesting since for a fixed quantifier depth only finitely many languages are definable, see *e.g.* [5]. To date, the situation with alternation depth is totally different: Only the very first level (*i.e.*, no alternation) is decidable [10, 23]; for all other levels no decidable characterizations are known. By a result of Thomas [28] the alternation hierarchy in first-order logic is tightly connected with the dot-depth hierarchy [3] or the Straubing-Thérien hierarchy [24, 26], depending on the presence or absence of the successor predicate. Some progress in the study of the dot-depth hierarchy and the Straubing-Thérien hierarchy was obtained by considering the half-levels. For example, the levels $1/2$ and $3/2$ in each of the two hierarchies are decidable [8, 19, 20]. The half levels also have a counterpart in the FO alternation hierarchy which is obtained by requiring existential quantifiers in the first block. Another point of view of the same hierarchy is to disallow universal quantifiers and to restrict the number of nested negations. Inside two-variable first-order logic, alternation depth is also a natural restriction. One difference to full first-order logic is that one cannot directly use prenex normal forms as a simple way of defining the alternation depth. Weil and the first author gave an effective algebraic characterization of the m^{th} level FO^2_m of this hierarchy. More precisely, they showed that it is possible to ascend the FO^2-alternation hierarchy using so-called Mal'cev products [15] which in this particular case preserve decidability. These Mal'cev products lead to the Trotter-Weil hierarchy for which many characterizations of FO^2 admit counterparts [12, 14, 16, 30]. An important tool in the study of the Trotter-Weil hierarchy are *rankers*. This combinatorial description was introduced by Schwentick, Thérien, and Vollmer [22] as *turtle programs* and was later refined by Weis and Immerman [31] to obtain a structure theorem for FO^2_m. Krebs and Straubing gave another decidable characterization of FO^2_m in terms of identities of omega-terms using completely different techniques [11]; their proof does not rely on rankers.

In this paper we consider the half-levels of the FO^2-alternation hierarchy. A language is hence definable in Σ^2_m if and only if it is definable in FO^2 without universal quantifiers and with at most m nested negations. It is easy to see that one can avoid negations of atomic predicates. By eliminating negations (using De Morgan's laws and $\neg\exists x\, \varphi \equiv \forall x\, \neg\varphi$), one can think of Σ^2_m as those FO^2-formulae φ such that every path of the parse tree of φ has at most m blocks of nested quantifiers and the outermost block is existential. The main contribution of this paper are omega-terms U_m and V_m such that an FO^2-definable language is Σ^2_m-definable if and only if its ordered syntactic monoid satisfies $U_m \leqslant V_m$. For a given regular language it is therefore decidable whether it is definable in Σ^2_m by first checking whether it is FO^2-definable and if so, then verifying $U_m \leqslant V_m$ for the ordered syntactic monoid. Moreover, for every FO^2-definable language L one can compute the smallest integer m such that L is definable in Σ^2_m. An important tool in the proof of our decidability criterion for Σ^2_m is a combinatorial characterization in terms of rankers. A corollary of this is the Weis-Immerman characterization of FO^2. On the other hand, there is no immediate connection between the decidability of FO^2_m and the decidability of Σ^2_m. A convenient intermediate step in the proofs is unary temporal logic $\mathrm{TL}[\mathsf{XF}, \mathsf{YP}]$ with strict future and past operators [7]. Table 1 summarizes the current state of the art (the parameter n denoting either the quantifier depth in FO or the operator depth in temporal logic).

2 Preliminaries

Languages and words. In this paper A is a finite alphabet, A^* is the set of finite words, and $A^+ = A^* \setminus \{\varepsilon\}$ are the nonempty finite words. A *language* is a subset of A^*. For a word $w = a_1 \cdots a_k$ with $a_i \in A$ the length of w is $|w| = k$, and the alphabet of w is $\mathrm{alph}(w) = \{a_1, \ldots, a_k\}$. A word $u = b_1 \cdots b_\ell$ with $b_i \in A$ is a *subword* of w if $w \in A^* b_1 \cdots A^* b_\ell A^*$. For integers p, q let $w[p;q] = a_p \cdots a_q$; if $p = q$, then we shorten this notation to $w[p]$. We say that p is an *a-position* if $w[p] = a$.

Rankers. A *ranker* is a nonempty word over $\{\mathsf{X}_a, \mathsf{Y}_a \mid a \in A\}$. The symbol X_a means neXt-a and is interpreted as an instruction of the form "go to the next a-position"; similarly, Y_a is for Yesterday-a and means "go to the previous a-position". For a word w and an integer p let

$$\mathsf{X}_a(w,p) = \min\{q > p \mid q \text{ is an } a\text{-position of } w\},$$
$$\mathsf{Y}_a(w,p) = \max\{q < p \mid q \text{ is an } a\text{-position of } w\}.$$

The minimum and maximum of the empty set are undefined. Inductively, for a ranker r and $\mathsf{Z} \in \{\mathsf{X}_a, \mathsf{Y}_a \mid a \in A\}$ let $\mathsf{Z}r(w,p) = r(w, \mathsf{Z}(w,p))$. In particular, rankers are processed from left to right. If either $\mathsf{Z}(w,p)$ or $r(w, \mathsf{Z}(w,p))$ is undefined, then $\mathsf{Z}r(w,p)$ is undefined. On words we let $\mathsf{X}_a r(w) = \mathsf{X}_a r(w,0)$ and $\mathsf{Y}_a r(w) = \mathsf{Y}_a r(w, |w|+1)$, *i.e.*, if the first modality of a ranker r is X_a, then r starts its execution in front of the word; symmetrically, if the first modality of r is Y_a, then r starts its execution after the end of the word. The *depth* of a ranker is its length as a word over the alphabet $\{\mathsf{X}_a, \mathsf{Y}_a \mid a \in A\}$. Let $R_{m,n}$ be the set of rankers with at most $m-1$ direction alternations and depth at most n; *i.e.*, rankers r with $|r| \leqslant n$ of the form $r = r_1 \cdots r_m$ such that $r_i \in \{\mathsf{X}_a \mid a \in A\}^*$ or $r_i \in \{\mathsf{Y}_a \mid a \in A\}^*$ for all i. We say that a ranker *ends with* an X-modality (respectively a Y-modality) if it is of the form $r\mathsf{X}_a$ (respectively $r\mathsf{Y}_a$) for some $a \in A$. For a set of rankers R and a word w let $R(w)$ consist of all rankers in R which are defined on w.

First-order logic. We consider first-order logic FO = FO[<] over finite words. The syntax of FO-formulae is

$$\top \mid \bot \mid \lambda(x) = a \mid x = y \mid x < y \mid \neg\varphi \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \exists x\, \varphi$$

where $a \in A$ is a letter, x and y are variables, and φ and ψ are formulae in FO. We consider universal quantifiers $\forall x\, \varphi$ as abbreviations for $\neg\exists x\, \neg\varphi$. The atomic formulae $\top$ and $\bot$ are *true* and *false*, respectively. Variables are interpreted as positions in the word model, and $\lambda(x) = a$ is true if x is an a-position. The semantics of the other constructs is as usual, in particular $\exists x\, \varphi$ means that there exist a position x which makes φ true, and $x < y$ means that position x is (strictly) smaller than position y. We write $\varphi(x_1, \ldots, x_\ell)$ for a formula φ if at most the variables x_i appear freely in φ; and we write $w, p_1, \ldots, p_\ell \models \varphi(x, \ldots, x_\ell)$ if φ is true over w when x_i is interpreted as p_i.

The *negation nesting* hierarchy within first-order logic is defined as follows. Let Σ_0 consist of disjunctions and conjunctions of atomic first-order formulae; and for $m \geqslant 1$ let Σ_m be the smallest class of first-order formulae containing φ and $\neg\varphi$ for all $\varphi \in \Sigma_{m-1}$ as well as $\varphi \vee \psi$, $\varphi \wedge \psi$, and $\exists x\, \varphi$ for all variables x and all $\varphi, \psi \in \Sigma_m$. Thus, when not counting negations of atomic formulae, then Σ_m for $m \geqslant 1$ contains the formulae with negation nesting depth is at most $m - 1$. Also note that, up to logical equivalence, our definition of Σ_m coincides with the more common definition in terms of formulae in prenex normal form with at most m blocks of quantifiers which start with an existential block. This can be seen by the usual procedure of renaming the variables and successively moving quantifiers outwards.

The two-variable fragment FO^2 of first-order logic uses (and reuses) only two distinct variables, say x and y. Combining FO^2 and Σ_m yields the fragment Σ^2_m. That is, we have $\varphi \in \Sigma^2_m$ if both $\varphi \in \Sigma_m$ and φ only uses the variables x and y. This also justifies the notation Σ^2_m which inherits the symbol as well as the subscript from Σ_m and from FO^2 the exponent. The further restriction of Σ^2_m to formulae of quantifier depth at most n yields the fragments $\Sigma^2_{m,n}$. The Boolean closure of Σ^2_m (respectively, $\Sigma^2_{m,n}$) is the m^{th} level of the *alternation hierarchy* FO^2_m (respectively, $\mathrm{FO}^2_{m,n}$) within FO^2.

Unary temporal logic. Unary temporal logic $\mathrm{TL}[\mathsf{XF}, \mathsf{YP}]$ consists of formulae built from atomic formulae $\top$ for *true*, $\bot$ for *false*, and a where $a \in A$ using Boolean connectives and modalities XF for neXt-Future and YP for Yesterday-Past. The semantics is given by the following two-variable formulae in one free variable: For a it is given by $\lambda(x) = a$, and we let $(\mathsf{XF}\varphi)(x) \equiv \exists y > x \colon \varphi(y)$ as well as $(\mathsf{YP}\varphi)(x) \equiv \exists y < x \colon \varphi(y)$; the remaining connectives are as usual. The formula $\varphi(y)$ in FO^2 is obtained from $\varphi(x)$ by interchanging x and y. On words without an interpreted variable let $w \models \top$, $w \not\models \bot$, and $w \not\models a$ and $\mathsf{XF}\varphi \equiv \mathsf{YP}\varphi \equiv \exists x\, \varphi(x)$; Boolean connectives are straightforward. Due to their identical definition, outermost YP-modalities can be replaced by XF without changing the semantics.

The negation nesting hierarchy of temporal logic is as follows. Let $\mathrm{TL}_0^+[\mathsf{XF},\mathsf{YP}]$ comprise atomic label formulae. For $m \geqslant 1$ let $\mathrm{TL}_m^+[\mathsf{XF},\mathsf{YP}]$ be the smallest class of formulae containing both φ and $\neg\varphi$ for $\mathrm{TL}_{m-1}^+[\mathsf{XF},\mathsf{YP}]$ as well as $\varphi \vee \psi$, $\varphi \wedge \psi$, $\mathsf{XF}\varphi$, and $\mathsf{YP}\varphi$ for $\varphi, \psi \in \mathrm{TL}_m^+[\mathsf{XF},\mathsf{YP}]$. Restricting the operator depth to at most n yields $\mathrm{TL}_{m,n}^+[\mathsf{XF},\mathsf{YP}]$, *i.e.*, on every path in the parse tree there are at most n nested modalities XF and YP.

For a first-order formula without free variables or a temporal logic formula φ let the *language defined* by φ be $L(\varphi) = \{w \in A^* \mid w \models \varphi\}$. Let $\mathcal{F}$ be a class of first-order or temporal formulae. A language L is *definable* in $\mathcal{F}$ if there exists $\varphi \in \mathcal{F}$ with $L(\varphi) = L$. We write $u \leqslant_{\mathcal{F}} v$ for words $u, v \in A^*$ if $v \models \varphi$ implies $u \models \varphi$ for all $\varphi \in \mathcal{F}$. For conciseness we use the notation $\leqslant_{m,n}^{\mathrm{FO}^2}$ instead $\leqslant_{\mathcal{F}}$ with $\mathcal{F} = \Sigma_{m,n}^2[<]$, and we write $\leqslant_{m,n}^{\mathrm{TL}}$ for $\leqslant_{\mathcal{G}}$ with $\mathcal{G} = \mathrm{TL}_{m,n}^+[\mathsf{XF},\mathsf{YP}]$.

3 Ranker Characterization of Σ_m^2

The next definition gives the main condition for the combinatorial characterization of $\Sigma_{m,n}^2$.

Definition 1. *Let $u, v \in A^*$ and $m, n \geqslant 0$. Let $u \leqslant_{m,n}^{\mathrm{WI}} v$ if either $m = 0$ or $n = 0$, or if $v \leqslant_{m-1,n}^{\mathrm{WI}} u$ and all of the following hold:*

1. *$R_{m,n}(v) \subseteq R_{m,n}(u)$,*
2. *$r(v) < s(v) \Rightarrow r(u) < s(u)$ and $r(v) \leqslant s(v) \Rightarrow r(u) \leqslant s(u)$ for all rankers r, s ending with an X-modality with $r \in R_{m,n}(v)$ and $s \in R_{m-1,n-1}(v)$,*
3. *$r(v) > s(v) \Rightarrow r(u) > s(u)$ and $r(v) \geqslant s(v) \Rightarrow r(u) \geqslant s(u)$ for all rankers r, s ending with a Y-modality with $r \in R_{m,n}(v)$ and $s \in R_{m-1,n-1}(v)$,*
4. *$r(v) < s(v) \Rightarrow r(u) < s(u)$ and $r(v) \leqslant s(v) \Rightarrow r(u) \leqslant s(u)$ for all rankers $r, s \in R_{m,n}(v)$ with $|r| + |s| < 2n$ such that r ends with an X-modality and s ends with a Y-modality.*

The relation $\leqslant_{m,n}^{\mathrm{WI}}$ is a preorder on A^*; the exponent is for "W*eis*-I*mmerman*" who introduced a similar condition for the fragment $\mathrm{FO}_{m,n}^2$; *cf.* [31]. We can now state our first main result.

Theorem 1. *Let $L \subseteq A^*$ and $m, n \geqslant 1$. The following conditions are equivalent:*

1. *L is definable in $\Sigma_{m,n}^2$.*
2. *L is definable in $\mathrm{TL}_{m,n}^+[\mathsf{XF},\mathsf{YP}]$.*
3. *L is an $\leqslant_{m,n}^{\mathrm{WI}}$-order ideal, i.e., $u \leqslant_{m,n}^{\mathrm{WI}} v$ and $v \in L$ implies $u \in L$.*

This result implies the characterization of Weis and Immerman [31]: Two words $u, v \in A^*$ model the same formulae of $\mathrm{FO}_{m,n}^2$ if and only if $R_{m,n}(u) = R_{m,n}(v)$ and $\mathrm{ord}(r(u), s(u)) = \mathrm{ord}(r(v), s(v))$ for all $r, s \in R_{m,n}(u)$ such that either $s \in R_{m,n-1}$

and r and s end with opposing directions or $s \in R_{m-1,n-1}$. Here, $\mathrm{ord}(p,q)$ is the *order-type*, *i.e.*, the unique element $\sim$ in the set $\{<,=,>\}$ such that $p \sim q$.

In the remainder of this section we prove Theorem 1. The translation from $\Sigma^2_{m,n}$ to $\mathrm{TL}^+_{m,n}[\mathsf{XF},\mathsf{YP}]$ is similar to a construction of Etessami, Vardi, and Wilke [7, proof of Theorem 1]. From temporal logic to rankers, we also have to take free variables into account. This step is the most technical one. Finally, we show that rankers and their order can be defined by suitable FO^2-sentences.

Lemma 1.1. *Let $m,n \geqslant 0$. Every $\Sigma^2_{m,n}$-definable language is $\mathrm{TL}^+_{m,n}[\mathsf{XF},\mathsf{YP}]$-definable.*

Proof. The construction is by induction on the structure of the formula; for the inductive step, we also have to take free variables into account. Let $\varphi(x,y) \in \Sigma^2_{m,n}$. We start by some normalizations on the structure of φ. We assume without restriction that on all paths in the parse tree of φ no two successive quantifiers bind the same variable. Therefore, by starting the construction with a sentence having this property, we can assume that the first quantifier on every path in the parse tree of φ binds the variable y, *i.e.*, when thinking of φ as a subformula of some sentence, then on the path to φ the previously bound variable is x.

Let $\tau \in \{<,=,>\}$ and $a \in A$. We show that there exists $\langle\varphi\rangle_{\tau,a}(x)$ in $\mathrm{TL}^+_{m,n}[\mathsf{XF},\mathsf{YP}]$ such that for all $u \in A^*$ and all positions p and q of u with $u[q] = a$ and $\mathrm{ord}(p,q) = \tau$ we have $u,p,q \models \varphi$ if and only if $u,p \models \langle\varphi\rangle_{\tau,a}$.

The construction is by induction on the structure of the formula. Let $\langle\top\rangle_{\tau,a}(x) \equiv \top$, $\langle\bot\rangle_{\tau,a}(x) \equiv \bot$ and for the other atomic formulae we set $\langle\lambda(x) = b\rangle_{\tau,a}(x) \equiv b$ and

$$\langle\lambda(y) = b\rangle_{\tau,a}(x) \;\equiv\; \begin{cases} \top & \text{if } a = b, \\ \bot & \text{otherwise,} \end{cases}$$

$$\langle x < y\rangle_{\tau,a}(x) \;\equiv\; \begin{cases} \top & \text{if } \tau \text{ is } <, \\ \bot & \text{otherwise.} \end{cases}$$

The formula $\langle y < x\rangle_{\tau,a}(x)$ is defined similarly. For the conjunction and the disjunction let $\langle\varphi \wedge \psi\rangle_{\tau,a}(x) \equiv \langle\varphi\rangle_{\tau,a}(x) \wedge \langle\psi\rangle_{\tau,a}(x)$ and $\langle\varphi \vee \psi\rangle_{\tau,a}(x) \equiv \langle\varphi\rangle_{\tau,a}(x) \vee \langle\psi\rangle_{\tau,a}(x)$, respectively. For negation we set $\langle\neg\varphi\rangle_{\tau,a}(x) \equiv \neg\langle\varphi\rangle_{\tau,a}(x)$. For existential quantification we let

$$\langle\exists y\, \varphi\rangle_{\tau,a}(x) \;\equiv\; \bigvee_{b \in A} b \wedge \big(\mathsf{YP}\langle\varphi\rangle_{<,b}(y) \vee \langle\varphi\rangle_{=,b}(y) \vee \mathsf{XF}\langle\varphi\rangle_{>,b}(y)\big).$$

In the construction of the formulae $\langle\varphi\rangle_{\tau',b}(y)$, the roles of x and y are interchanged. Note that due to our normalization at the beginning, we do not have to handle quantification over x.

We now define $\varphi' \in \mathrm{TL}^+_{m,n}[\mathsf{XF},\mathsf{YP}]$ inductively for sentences $\varphi \in \Sigma^2_{m,n}$. Boolean connectives are straightforward: $(\varphi \wedge \psi)' \equiv \varphi' \wedge \psi'$, $(\varphi \vee \psi)' \equiv \varphi' \vee \psi'$ and $(\neg\varphi)' \equiv \neg\varphi'$. For quantification we set $(\exists y\colon \varphi)' \equiv \mathsf{XF}(\langle\varphi\rangle_{\tau,a}(y))$ where $\tau \in \{<,=,>\}$ and $a \in A$ are arbitrary. Hence for every sentence $\varphi \in \Sigma^2_{m,n}$ there exists $\varphi' \in \mathrm{TL}^+_{m,n}[\mathsf{XF},\mathsf{YP}]$ such that $L(\varphi) = L(\varphi')$. $\square$

We now introduce a relation $\preccurlyeq^{\mathrm{WI}}_{m,n}$ which resembles $\leqslant^{\mathrm{WI}}_{m,n}$ for free variables. For $u,v \in A^*$ and non-negative integers x,x' let $(u,x') \preccurlyeq^{\mathrm{WI}}_{m,n} (v,x)$ if $u \leqslant^{\mathrm{WI}}_{m,n} v$ and either

$x = x' = 0$ or $v[x] = u[x']$, and if further either $m = 0$, or $n = 0$, or all of the following hold:

1. $s(v) < x \Rightarrow s(u) < x'$ if $s \in R_{m,n}(v)$ ends with an X-modality or if $s \in R_{m-1,n}(v)$,
2. $s(v) \leqslant x \Rightarrow s(u) \leqslant x'$ if $s \in R_{m-1,n}(v)$,
3. $s(v) > x \Rightarrow s(u) > x'$ if $s \in R_{m,n}(v)$ ends with a Y-modality or if $s \in R_{m-1,n}(v)$,
4. $s(v) \geqslant x \Rightarrow s(u) \geqslant x'$ if $s \in R_{m-1,n}(v)$.

The idea is that $x = x' = 0$ encodes the case without free variables. The following lemma shows that the roles of u and v can be interchanged by investing one negation.

Lemma 1.2. *Let $u, v \in A^*$ and $m, n \geqslant 1$. If $(u, x') \preccurlyeq^{\text{WI}}_{m,n} (v, x)$, then $(v, x) \preccurlyeq^{\text{WI}}_{m-1,n} (u, x')$.*

Proof. We may assume $m \geqslant 2$ and $n \geqslant 1$ since the claim is trivial otherwise. The alphabetic condition is clear and $v \leqslant^{\text{WI}}_{m-1,n} u$ is a requirement of $u \leqslant^{\text{WI}}_{m,n} v$. Suppose $s(u) < x'$ for some $s \in R_{m-1,n}(u)$ which ends with an X-modality or some $s \in R_{m-2,n}(u)$. We have to show $s(v) < x$. Suppose $s(v) \geqslant x$ for the sake of contradiction. By condition (4) for $(u, x') \preccurlyeq^{\text{WI}}_{m,n} (v, x)$ we obtain $s(u) \geqslant x'$, contradicting the assumption $s(u) < x'$. A similar reasoning applies to each of the remaining three conditions. □

Lemma 1.3. *Let $u, v \in A^*$ and $m, n \geqslant 1$. If $(u, x') \preccurlyeq^{\text{WI}}_{m,n} (v, x)$, then for all $1 \leqslant y \leqslant |v|$ there exists $1 \leqslant y' \leqslant |u|$ with $\operatorname{ord}(x, y) = \operatorname{ord}(x', y')$ and $(u, y') \preccurlyeq^{\text{WI}}_{m,n-1} (v, y)$.*

Proof. For $y = x$ simply take $y' = x'$. By left-right symmetry it suffices to consider the case $y > x$. Let $a = v[y]$ and

$$R_{right} = \{r' \mid r'\mathsf{Y}_a \in R_{m,n}(v) \text{ and } r'\mathsf{Y}_a(v) \geqslant y\},$$
$$R_{left} = \{r' \mid r'\mathsf{X}_a \in R_{m-1,n-1}(v) \text{ and } r'\mathsf{X}_a(v) \geqslant y\}.$$

Note that $R_{right}\,\mathsf{Y}_a \cup R_{left}\,\mathsf{X}_a \subseteq R_{m,n}(v) \subseteq R_{m,n}(u)$. Let $r \in R_{right}\,\mathsf{Y}_a \cup R_{left}\,\mathsf{X}_a$ be such that $r(u)$ is minimal. We claim that we can choose $y' = r(u)$. By $r(v) \geqslant y > x$ and condition (3) in the definition of $\preccurlyeq^{\text{WI}}_{m,n}$ we conclude $y' = r(u) > x'$.

Next, we show $(u, y') \preccurlyeq^{\text{WI}}_{m,n-1} (v, y)$. The alphabetic condition is clear. By $R^{\mathsf{Z}}_{m,n}$ we denote the rankers in $R_{m,n}$ ending with a Z-modality for $\mathsf{Z} \in \{\mathsf{X}, \mathsf{Y}\}$. "Condition (1)": Let $s \in R^{\mathsf{X}}_{m,n-1}(v) \cup R_{m-1,n-1}(v)$ with $s(v) < y$. In particular, we have $s(v) < r(v)$. By Definition 1, $u \leqslant^{\text{WI}}_{m,n} v$ yields $s(u) < r(u) = y'$. (If $s \in R^{\mathsf{X}}_{m,n-1}$ and $r \in R^{\mathsf{Y}}_{m,n}$, then condition (4) of Definition 1 is applied. If $s \in R^{\mathsf{X}}_{m,n-1}$ and $r \in R^{\mathsf{X}}_{m-1,n-1}$, then condition (2) is applied. If $s \in R^{\mathsf{Y}}_{m-1,n-1}$ and $r \in R^{\mathsf{Y}}_{m,n}$, then condition (3) is applied. If $s \in R^{\mathsf{Y}}_{m-1,n-1}$ and $r \in R^{\mathsf{X}}_{m-1,n-1}$, then condition (4) for $v \leqslant^{\text{WI}}_{m-1,n} u$ is applied.) Verifying condition (2) is similar. "Condition (3)": Let $s \in R^{\mathsf{Y}}_{m,n-1}(v) \cup R_{m-1,n-1}(v)$ and $s(v) > y$. Since $s \in R_{right}$, we see that $s(u) > s\mathsf{Y}_a(u) \geqslant r(u) = y'$ by choice of r. "Condition (4)": Let $s \in R_{m-1,n-1}(v)$ and $s(v) \geqslant y$. First suppose $s = s'\mathsf{X}_a$. Then $s' \in R_{left}$ and thus $s(u) = s'\mathsf{X}_a(u) \geqslant r(u) = y'$ by choice of r. Finally suppose $s = s'\mathsf{Y}_a$. Then $s' \in R_{right}$ and thus $s(u) = s'\mathsf{Y}_a(u) \geqslant r(u) = y'$ by choice of r. □

Using Lemmas 1.2 and 1.3, a straightforward induction shows that $\mathrm{TL}^{+}_{m,n}[\mathsf{XF}, \mathsf{YP}]$-definable languages are $\leqslant^{\text{WI}}_{m,n}$-order ideals. This is recorded in the following lemma.

Lemma 1.4. *Let $u, v \in A^*$ and $m, n \geqslant 0$. If $u \leqslant^{\text{WI}}_{m,n} v$, then $u \leqslant^{\text{TL}}_{m,n} v$.*

Proof. We show that if $(u, x') \preccurlyeq^{\text{WI}}_{m,n} (v, x)$, then $v, x \models \varphi$ implies $u, x' \models \varphi$ for all formulae $\varphi \in \text{TL}^+_{m,n}[\mathsf{XF}, \mathsf{YP}]$. The proof of this claim is by structural induction. For atomic formulae this follows by the alphabetic condition in the definition of $\preccurlyeq^{\text{WI}}_{m,n}$. In particular we may assume $m, n \geqslant 1$. For Boolean connectives the claim follows by induction; in the case of negation this relies on Lemma 1.2 and $\varphi \in \text{TL}^+_{m-1,n}[\mathsf{XF}, \mathsf{YP}]$ whenever $\neg\varphi \in \text{TL}^+_{m,n}[\mathsf{XF}, \mathsf{YP}]$. Suppose $v, x \models \mathsf{XF}\varphi$. Then there exists a position $y > x$ of v such that $v, y \models \varphi$. By Lemma 1.3 there exists a position $y' > x'$ of u such that $(u, y') \preccurlyeq^{\text{WI}}_{m,n-1} (v, y)$. Because we have $\varphi \in \text{TL}^+_{m,n-1}[\mathsf{XF}, \mathsf{YP}]$, induction yields $u, y' \models \varphi$ and finally $u, x' \models \mathsf{XF}\varphi$. The remaining case $\mathsf{YP}\varphi$ is symmetric to $\mathsf{XF}\varphi$. This concludes the proof of the claim.

Suppose now $v \models \varphi$ for some $\varphi \in \text{TL}^+_{m,n}[\mathsf{XF}, \mathsf{YP}]$. Replacing outermost YP-modalities by XF yields a formula which defines the same language. We may therefore assume without loss of generality that $v \models \varphi$ if and only if $v, 0 \models \varphi$. If $u \leqslant^{\text{WI}}_{m,n} v$, then $(u, 0) \preccurlyeq^{\text{WI}}_{m,n} (v, 0)$ and the claim thus yields $u, 0 \models \varphi$. This shows $u \leqslant^{\text{TL}}_{m,n} v$. □

We now give two-variable formulae for order comparisons with ranker positions. These formulae show that a direction alternation in a ranker can be covered by a negation in first-order logic.

Lemma 1.5. *Let $r \in R_{m,n}$. If r ends with an X-modality, then there exist formulae $\langle x \geqslant r \rangle$ and $\langle x > r \rangle$ in $\Sigma^2_{m,n}$ such that for all words u with $r(u)$ being defined and for all positions i of u we have*

$$u, i \models \langle x \geqslant r \rangle \quad \text{if and only if} \quad i \geqslant r(u),$$
$$u, i \models \langle x > r \rangle \quad \text{if and only if} \quad i > r(u).$$

Symmetrically, if r ends with a Y-modality, then there exist formulae $\langle x \leqslant r \rangle$ and $\langle x < r \rangle$ in $\Sigma^2_{m,n}$ such that for all words u with $r(u)$ being defined and for all positions i of u we have

$$u, i \models \langle x \leqslant r \rangle \quad \text{if and only if} \quad i \leqslant r(u),$$
$$u, i \models \langle x < r \rangle \quad \text{if and only if} \quad i < r(u).$$

Proof. It suffices to specify $\langle x > r \rangle$ and $\langle x \geqslant r \rangle$ for rankers ending on X. The formulae $\langle x < r \rangle$ and $\langle x \leqslant r \rangle$ for rankers ending on Y are then left-right symmetric. The proof is by induction on the depth of r. Let

$$\langle x > \mathsf{X}_a \rangle \equiv \exists y < x \colon \lambda(y) = a, \qquad \langle x \geqslant \mathsf{X}_a \rangle \equiv \exists y \leqslant x \colon \lambda(y) = a,$$
$$\langle x > r\mathsf{X}_a \rangle \equiv \exists y < x \,\big(\lambda(y) = a \wedge \langle y > r \rangle\big), \; \langle x \geqslant r\mathsf{X}_a \rangle \equiv \exists y \leqslant x \,\big(\lambda(y) = a \wedge \langle y > r \rangle\big).$$

where r is a ranker. Here, $\langle y > r \rangle$ is obtained by induction if r ends on an X-modality; otherwise we let $\langle y > r \rangle \equiv \neg\langle y \leqslant r \rangle$ with $\langle y \leqslant r \rangle$ obtained by induction. As usual, the formula $\langle y > r \rangle$ with free variable y is obtained by interchanging x and y. □

The formulae in Lemma 1.5 yield suitable two-variable sentences for rankers and for order comparisons of rankers.

Lemma 1.6. *Let $u, v \in A^*$ and $m, n \geqslant 0$ be integers. If $u \leqslant^{\mathrm{FO}^2}_{m,n} v$, then $u \leqslant^{\mathrm{WI}}_{m,n} v$.*

Proof. The proof is by induction on m with the trivial base case $m = 0$. Suppose $u \leqslant^{\mathrm{FO}^2}_{m,n} v$. In particular $v \leqslant^{\mathrm{FO}^2}_{m-1,n} u$ and hence $v \leqslant^{\mathrm{WI}}_{m-1,n} u$ by induction. We show conditions (1) to (4) of Definition 1 one after another. The proof makes extensive use of Lemma 1.5.

"Condition (1)": Suppose $r \in R_{m,n}$ is defined on v but not on u. Let $r = r'\mathsf{Z}r''$ with $\mathsf{Z} \in \{\mathsf{X}_a, \mathsf{Y}_a\}$ for some $a \in A$ such that r' is the longest prefix of r which is defined on u. Note that $\mathrm{alph}(v) \subseteq \mathrm{alph}(u)$ and thus $|r'| \geqslant 1$. Consider the formula

$$\langle r'\mathsf{Z}\rangle \;\equiv\; \exists x\colon \lambda(x) = a \wedge \begin{cases} \langle x > r'\rangle & \text{if } \mathsf{Z} = \mathsf{X}_a, \\ \langle x < r'\rangle & \text{if } \mathsf{Z} = \mathsf{Y}_a. \end{cases}$$

In both cases $\langle r'\mathsf{Z}\rangle \in \Sigma^2_{m,n}$, and $\langle r'\mathsf{Z}\rangle$ is true on $w \in A^*$ if and only if $r'\mathsf{Z}(w)$ is defined. In particular, $r'\mathsf{Z}$ is defined on u which contradicts the definition of r'. Therefore, $r(u)$ is defined. This shows $R_{m,n}(v) \subseteq R_{m,n}(u)$.

"Condition (2)": Consider rankers r, s with $r \in R_{m,n}(v)$ and $s \in R_{m-1,n-1}(v)$ which end with an X-modality. Let $r = r'\mathsf{X}_a$ for $a \in A$ and let $\lesssim \in \{<, \leqslant\}$. The formula

$$\langle r \lesssim s\rangle \;\equiv\; \exists x\colon \lambda(x) = a \wedge \langle x > r'\rangle \wedge \langle x \lesssim s\rangle$$

is in $\Sigma^2_{m,n}$. Here we set $\langle x > r'\rangle = \top$ whenever r' is empty. Moreover $\langle r \lesssim s\rangle$ is true on $w \in A^*$ if and only if $r(w) \lesssim s(w)$. Hence $r(v) < s(v)$ implies $r(u) < s(u)$, and $r(v) \leqslant s(v)$ implies $r(u) \leqslant s(u)$.

"Condition (3)": This is symmetric to property (2).

"Condition (4)": Consider rankers $r, s \in R_{m,n}(v)$ with $|r| + |s| < 2n$ such that r ends with an X-modality and s ends with a Y-modality. Let $\lesssim \in \{<, \leqslant\}$ and $\gtrsim$ be its inverse relation. Let

$$\langle r \lesssim s\rangle \;\equiv\; \exists x\colon \lambda(x) = a \wedge \begin{cases} \langle x > r'\rangle \wedge \langle x \lesssim s\rangle & \text{if } r = r'\mathsf{X}_a \text{ and } s \in R_{m,n-1}(v), \\ \langle x < s'\rangle \wedge \langle x \gtrsim r\rangle & \text{if } r \in R_{m,n-1}(v) \text{ and } s = s'\mathsf{Y}_a. \end{cases}$$

In both cases $\langle r \lesssim s\rangle \in \Sigma^2_{m,n}$ and $\langle r \lesssim s\rangle$ is true on $w \in A^*$ if and only if $r(w) \lesssim s(w)$. Therefore $r(v) < s(v)$ implies $r(u) < s(u)$, and $r(v) \leqslant s(v)$ implies $r(u) \leqslant s(u)$. □

Theorem 1. "(1) ⇒ (2)": This is Lemma 1.1.

"(2) ⇒ (3)": Let L be definable in $\mathrm{TL}^+_{m,n}[\mathsf{XF}, \mathsf{YP}]$. Suppose $u \leqslant^{\mathrm{WI}}_{m,n} v$ and $v \in L$. Lemma 1.4 yields $u \leqslant^{\mathrm{TL}}_{m,n} v$ and consequently $u \in L$. This shows that L is a $\leqslant^{\mathrm{WI}}_{m,n}$-order ideal.

"(3) ⇒ (1)": Let L be a $\leqslant^{\mathrm{WI}}_{m,n}$-order ideal. Suppose $u \leqslant^{\mathrm{FO}^2}_{m,n} v$ and $v \in L$. By Lemma 1.6 we see $u \leqslant^{\mathrm{WI}}_{m,n} v$ and thus $u \in L$. Therefore L is a $\leqslant^{\mathrm{FO}^2}_{m,n}$-order ideal. For a word u let L_u be the intersection of all $\Sigma^2_{m,n}$-definable languages containing u; this intersection is finite since up to equivalence there are only finitely many formulae in $\Sigma^2_{m,n}$. Moreover if $u \in L$, then we have $L_u \subseteq L$ and thus $L = \bigcup_{u \in L} L_u$; this union is also finite since there are only finitely many languages of the form L_u. □

4 Decidability of Σ^2_m

In this section we give a decidable algebraic characterization of languages definable in Σ^2_m. We start by introducing the concepts necessary for this result.

An *ordered monoid* $(M, \leqslant)$ is a monoid M equipped with a partial order $\leqslant$ which is compatible with multiplication in M; *i.e.*, $x \leqslant x'$ and $y \leqslant y'$ implies $xy \leqslant x'y'$. Every monoid can be considered as an ordered monoid when using the identity relation as order. An *order ideal* of $\leqslant$ is a subset $I \subseteq M$ such that $y \leqslant x$ and $x \in I$ implies $y \in I$. A language $L \subseteq A^*$ is *recognized* by a homomorphism $h\colon A^* \to M$ to an ordered monoid $(M, \leqslant)$ if $L = h^{-1}(I)$ for some $\leqslant$-order ideal I. A monoid M recognizes a language L if there exists a homomorphism $h\colon A \to M$ which recognizes L. The *syntactic preorder* $\leqslant_L$ on words is defined as follows: We set $u \leqslant_L v$ for $u, v \in A^*$ if $pvq \in L$ implies $puq \in L$ for all $p, q \in A^*$. We write $u \equiv_L v$ if both $u \leqslant_L v$ and $v \leqslant_L u$. The *syntactic monoid* of L is $A^*/{\equiv_L}$; it is the unique minimal recognizer of L and it is effectively computable from any reasonable presentation of given a regular language, see e.g. [18]. The syntactic preorder induces a partial order on $\equiv_L$-classes such that $A^*/{\equiv_L}$ becomes an ordered monoid. The *syntactic homomorphism* is $h_L\colon A^* \to A^*/{\equiv_L}$ is the natural quotient map.

Green's relations are an important tool in the study of finite monoids. For $x, y \in M$ let $x \leqslant_{\mathcal{R}} y$ if $xM \subseteq yM$, and let $x \leqslant_{\mathcal{L}} y$ if $Mx \subseteq My$. We write $x \mathrel{\mathcal{R}} y$ if both $x \leqslant_{\mathcal{R}} y$ and $y \leqslant_{\mathcal{R}} x$; and we set $x <_{\mathcal{R}} y$ if $x \leqslant_{\mathcal{R}} y$ and $y \not\leqslant_{\mathcal{R}} x$. The relations $\mathcal{L}$ and $<_{\mathcal{L}}$ are defined similarly. An element x in a monoid M is idempotent if $x^2 = x$. For every finite monoid M there exists an integer $\omega_M \geqslant 1$ such that x^{ω_M} is the unique idempotent power generated by $x \in M$. If the reference to M is clear from the context, we simply write ω instead of ω_M. Classes of finite (ordered) monoids are often described by identities of omega-terms. We define *omega-terms* over a set of variables Ω inductively: Every variable $x \in \Omega$ is an omega-term, and if u and v are omega-terms, then so are uv and u^ω. Here, ω is considered as a formal symbol instead of a fixed integer. Every mapping $h\colon \Omega \to M$ to a finite monoid M uniquely extends to omega-terms by setting $h(uv) = h(u)h(v)$ and $h(u^\omega) = h(u)^{\omega_M}$. An ordered monoid $(M, \leqslant)$ *satisfies* the inequality $U \leqslant V$ of omega-terms U, V if $h(U) \leqslant h(V)$ for all mappings $h\colon \Omega \to M$. It satisfies $U = V$ if it satisfies both $U \leqslant V$ and $V \leqslant U$.

An important class of monoids is **DA** which will serve as an upper bound for our algebraic characterization. Let **DA** be comprised of all finite monoids M satisfying $(xyz)^\omega y (xyz)^\omega = (xyz)^\omega$. Suppose $M \in \mathbf{DA}$ and let $u, v, a \in M$. If $v \mathrel{\mathcal{R}} u \mathrel{\mathcal{R}} ua$, then $v \mathrel{\mathcal{R}} va$; and symmetrically, if $v \mathrel{\mathcal{L}} u \mathrel{\mathcal{L}} au$, then $v \mathrel{\mathcal{L}} av$, see *e.g.* [12, Lemma 1].

As we shall see in this section, the following sequences of omega-terms U_m and V_m characterize Σ^2_m for monoids within **DA**. Let $U_1 = z$, let $V_1 = 1$, and let

$$U_m = (U_{m-1} x_m)^\omega U_{m-1} (y_m U_{m-1})^\omega$$
$$V_m = (U_{m-1} x_m)^\omega V_{m-1} (y_m U_{m-1})^\omega$$

for $m \geqslant 2$ and variables $x_2, y_2, \ldots, x_m, y_m, z$. We can now state the second main result.

Theorem 2. *Let $L \subseteq A^*$ be definable in FO^2 and let $m \geqslant 1$. Then L is definable in Σ^2_m if and only if the ordered syntactic monoid of L satisfies $U_m \leqslant V_m$.*

Thérien and Wilke [27] have shown that a language is definable in FO^2 if and only if its syntactic monoid is in **DA**. Since the syntactic monoid of a regular language (given for example as a finite automaton or a first-order sentence) is effectively computable, the above characterization is decidable.

Corollary 2.1. *It is decidable whether a given regular language is definable in* Σ_m^2. □

Note that the equivalence in Theorem 2 does not hold for arbitrary regular languages; for example the ordered syntactic monoid of the language $A^* \setminus A^*aaA^*$ over the alphabet $A = \{a, b\}$ satisfies the inequality $U_m \leqslant V_m$ for all $m \geqslant 2$ but it is not FO^2-definable. The remainder of this section is devoted to the proof of Theorem 2. It relies on a combination of algebraic properties of monoids in **DA** with combinatorial properties of Σ_m^2, leading to a factorization which enables induction on the parameter m.

We start with the combinatorics. The following two lemmas describe an important relativization technique inside FO^2. The first lemma restricts the interpretation of formulae to the prefix before (respectively to the suffix after) the first a-position of a word. Its left-right dual Lemma 2.2 gives restriction with respect to the last a-position. The second lemma relativizes to the factor between a crossing of the first a-position and the last b-position. In both situations we pay special attention to the parameter m.

Lemma 2.1. *Let* $\varphi \in \Sigma_{m,n}^2$ *for* $m, n \geqslant 0$, *and let* $a \in A$. *Then there exist formulae* $\langle\varphi\rangle_{<\mathsf{X}a}$ *in* $\Sigma_{m+1,n+1}^2$ *and* $\langle\varphi\rangle_{>\mathsf{X}a}$ *in* $\Sigma_{m,n+1}^2$ *such that for all words* $u = u_1 a u_2$ *with* $a \notin \mathrm{alph}(u_1)$:

$u, p, q \models \langle\varphi\rangle_{<\mathsf{X}a}$ *if and only if* $u_1, p, q \models \varphi$ *for all* $1 \leqslant p, q \leqslant |u_1|$,

$u, p, q \models \langle\varphi\rangle_{>\mathsf{X}a}$ *if and only if* $u_2, p - |u_1 a|, q - |u_1 a| \models \varphi$ *for all* $|u_1 a| < p, q \leqslant |u|$.

Proof. The formulae $\langle\varphi\rangle_{<\mathsf{X}a}$ and $\langle\varphi\rangle_{>\mathsf{X}a}$ relativize the interpretation of the formula by restricting the evaluation of quantifiers to the factor before (respectively after) the first occurrence of the letter a (*i.e.*, the position reached by the ranker X_a). We first give the inductive construction for $\langle\varphi\rangle_{<\mathsf{X}a}$. Let $\langle\varphi\rangle_{<\mathsf{X}a} \equiv \varphi$ if φ is an atomic formula. For conjunction let $\langle\varphi \wedge \psi\rangle_{<\mathsf{X}a} \equiv \langle\varphi\rangle_{<\mathsf{X}a} \wedge \langle\psi\rangle_{<\mathsf{X}a}$, for disjunction let $\langle\varphi \vee \psi\rangle_{<\mathsf{X}a} \equiv \langle\varphi\rangle_{<\mathsf{X}a} \vee \langle\psi\rangle_{<\mathsf{X}a}$, and for negation let $\langle\neg\varphi\rangle_{<\mathsf{X}a} \equiv \neg\langle\varphi\rangle_{<\mathsf{X}a}$. For existential quantification let

$$\langle\exists x\, \varphi\rangle_{<\mathsf{X}a} \;\equiv\; \exists x \left(\neg(\exists y \leqslant x \colon \lambda(y) = a) \wedge \langle\varphi\rangle_{<\mathsf{X}a}\right).$$

As usual, swapping the variables x and y yields the corresponding constructions for y. We now construct $\langle\varphi\rangle_{>\mathsf{X}a}$. Let $\langle\varphi\rangle_{>\mathsf{X}a} \equiv \varphi$ if φ is an atomic formula. Boolean combinations are as above and for existential quantification let

$$\langle\exists x\, \varphi\rangle_{>\mathsf{X}a} \;\equiv\; \exists x \left((\exists y < x \colon \lambda(y) = a) \wedge \langle\varphi\rangle_{>\mathsf{X}a}\right).$$

Again, the constructions for y are dual. □

The following is the left-right symmetric version of Lemma 2.1 and is thus admitted without proof.

Lemma 2.2. *Let* $\varphi \in \Sigma_{m,n}^2$ *for* $m, n \geqslant 0$, *and let* $a \in A$. *Then there exist formulae* $\langle\varphi\rangle_{<\mathsf{Y}a}$ *in* $\Sigma_{m,n+1}^2$ *and* $\langle\varphi\rangle_{>\mathsf{Y}a}$ *in* $\Sigma_{m+1,n+1}^2$ *such that for all words* $u = u_1 a u_2$ *with* $a \notin \mathrm{alph}(u_2)$:

$u, p, q \models \langle\varphi\rangle_{<\mathsf{Y}a}$ *if and only if* $u_1, p, q \models \varphi$ *for all* $1 \leqslant p, q \leqslant |u_1|$,
$u, p, q \models \langle\varphi\rangle_{>\mathsf{Y}a}$ *if and only if* $u_2, p - |u_1 a|, q - |u_1 a| \models \varphi$ *for all* $|u_1 a| < p, q \leqslant |u|$.

$\square$

Lemma 2.3. *Let $\varphi \in \Sigma^2_{m,n}$ for $m, n \geqslant 0$, and let $a, b \in A$. Then there exists a formula $\langle\varphi\rangle_{(\mathsf{Y}b;\mathsf{X}a)}$ in $\Sigma^2_{m+1,n+1}$ such that for all words $u = u_1 b u_2 a u_3$ with $b \notin \mathrm{alph}(u_2 a u_3)$ and $a \notin \mathrm{alph}(u_1 b u_2)$, and for all $|u_1 b| < p, q \leqslant |u_1 b u_2|$ we have:*

$$u, p, q \models \langle\varphi\rangle_{(\mathsf{Y}b;\mathsf{X}a)} \quad \textit{if and only if} \quad u_2, p - |u_1 b|, q - |u_1 b| \models \varphi.$$

Proof. We construct $\langle\varphi\rangle_{(\mathsf{Y}b;\mathsf{X}a)}$ which restricts the interpretation of φ to the factor of the model which is between the last b and the first a. Atomic formulae and Boolean combinations are straightforward. For existential quantification let

$$\langle\exists x\, \varphi\rangle_{(\mathsf{Y}b;\mathsf{X}a)} \;\equiv\; \exists x \left(\neg(\exists y \leqslant x\colon \lambda(y) = a) \wedge \neg(\exists y \geqslant x\colon \lambda(y) = b) \wedge \langle\varphi\rangle_{(\mathsf{Y}b;\mathsf{X}a)}\right).$$

Quantifications over y are dual. $\square$

Let $h\colon A^* \to M$ be a homomorphism. For a word $u \in A^*$ the *$\mathcal{L}$-factorization* is the unique factorization $u = s_0 a_1 \cdots s_{\ell-1} a_\ell s_\ell$ with $s_i \in A^*$ and so-called *markers* $a_i \in A$ such that $h(s_\ell) \mathrel{\mathcal{L}} 1$ and $h(s_i a_{i+1} \cdots s_{\ell-1} a_\ell s_\ell) >_{\mathcal{L}} h(a_i s_i \cdots a_\ell s_\ell) \mathrel{\mathcal{L}} h(s_{i-1} a_i \cdots s_{\ell-1} a_\ell s_\ell)$ for all i. In particular $\ell < |M|$. If $M \in \mathbf{DA}$, then $a_i \notin \mathrm{alph}(s_i)$. Let $D_{\mathcal{L}}(u)$ consist of the positions of the markers, *i.e.*, let $D_{\mathcal{L}}(u) = \{|s_0 a_1 \cdots s_{i-1} a_i| \geqslant 1 \mid 1 \leqslant i \leqslant \ell\}$. The $\mathcal{R}$-factorization is defined left-right symmetrically. The set $D_{\mathcal{R}}(u)$ consists of all positions p of u such that $h(u[1;p-1]) >_{\mathcal{R}} h(u[1;p])$. The following lemma combines the $\mathcal{R}$- and the $\mathcal{L}$-factorization for monoids in $\mathbf{DA}$.

Lemma 2.4. *Let $h\colon A^* \to M$ be a homomorphism with $M \in \mathbf{DA}$. Let $u, v \in A^*$ with $u \leqslant^{\mathrm{FO}^2}_{2,2|M|} v$. Then there exist factorizations $u = s_0 a_1 \cdots s_{\ell-1} a_\ell s_\ell$ and $v = t_0 a_1 \cdots t_{\ell-1} a_\ell t_\ell$ with $\ell < 2|M|$, $a_i \in A$ and $s_i, t_i \in A^*$ such that $h(s_0) \mathrel{\mathcal{R}} 1$ and $h(s_\ell) \mathrel{\mathcal{L}} 1$ and the following properties hold for all $1 \leqslant i \leqslant \ell$:*

1. *$h(t_0 a_1 \cdots t_{i-1} a_i) \mathrel{\mathcal{R}} h(t_0 a_1 \cdots t_{i-1} a_i s_i)$,*
2. *$h(a_i s_i \cdots a_\ell s_\ell) \mathrel{\mathcal{L}} h(s_{i-1} a_i \cdots a_\ell s_\ell)$.*

Moreover, for every $1 \leqslant i \leqslant \ell$ there exists a ranker r in

$$\{\mathsf{X}_{b_1} \cdots \mathsf{X}_{b_k}, \mathsf{Y}_{b_k} \cdots \mathsf{Y}_{b_1} \mid b_1 \cdots b_k \textit{ is a subword of } a_1 \cdots a_\ell \textit{ for } b_i \in A\}$$

with $r(u) = |s_0 a_1 \cdots s_{i-1} a_i|$ and $r(v) = |t_0 a_1 \cdots t_{i-1} a_i|$.

Proof. Loading the induction hypothesis, we allow for an additional prefix p of v; that is, we consider words u and pv but the factorizations are for u and v only. The only difference is that we require $h(p t_0 a_1 \cdots t_{i-1} a_i) \mathrel{\mathcal{R}} h(p t_0 a_1 \cdots t_{i-1} a_i s_i)$ instead of property (1). Note that the suffix at the end is s_i and not t_i. The assumption is $u \leqslant^{\mathrm{FO}^2}_{2,n} v$ for $n = |D_{\mathcal{R}}(pv) \setminus D_{\mathcal{R}}(p)| + |D_{\mathcal{L}}(u)| + 1$ and the proof proceeds by induction on $|D_{\mathcal{R}}(pv) \setminus D_{\mathcal{R}}(p)|$. Let $u = s'_0 c_1 \cdots s'_{\ell'-1} c_{\ell'} s'_{\ell'}$ and $v = t'_0 c_1 \cdots t'_{\ell'-1} c_{\ell'} t'_{\ell'}$ where the factorization of u is the $\mathcal{L}$-factorization (in particular $c_i \notin \mathrm{alph}(s'_i)$) and where $c_i \notin \mathrm{alph}(t'_i)$ for all i. This factorization of v exists because by assumption u and v agree on subwords of length ℓ'.

First suppose $D_{\mathcal{R}}(p) = D_{\mathcal{R}}(pv)$. In this case $h(p) \mathrel{\mathcal{R}} h(pv)$ and thus $h(p) \mathrel{\mathcal{R}} h(px)$ for all $x \in B^*$ where $B = \mathrm{alph}(v)$. In particular $h(p t'_0 c_1 \cdots t'_{i-1} c_i) \mathrel{\mathcal{R}} h(p t'_0 c_1 \cdots t'_{i-1} c_i s'_i)$

because $\mathrm{alph}(u) = B$. Hence, setting $a_i = c_i$, $s_i = s'_i$, and $t_i = t'_i$ yields a factorization with the desired properties.

Let now $D_{\mathcal{R}}(p) \subsetneq D_{\mathcal{R}}(pv)$. Let sa with $s \in A^*$ and $a \in A$ be the shortest prefix of u such that $h(p) \mathrel{\mathcal{R}} h(ps) >_{\mathcal{R}} h(psa)$. Such a prefix exists since $\mathrm{alph}(u) = \mathrm{alph}(v)$. We have $|sa| = \mathsf{X}_a(u)$ by $M \in \mathbf{DA}$. Let ta be the prefix of v satisfying $|ta| = \mathsf{X}_a(v)$. By Lemma 2.1 we have $\mathrm{alph}(t) \subseteq \mathrm{alph}(s)$. Next, let k be maximal such that $s'_0 c_1 \cdots s'_{k-1} c_k$ is a prefix of s. By Theorem 1 and Definition 1 (4), the index k is also maximal such that $t'_0 c_1 \cdots t'_{k-1} c_k$ is a prefix of t. Let $a_i = c_i$ for $i \in \{1, \ldots, k\}$, let $s_i = s'_i$ and $t_i = t'_i$ for $i \in \{0, \ldots, k-1\}$, and let s_k and t_k such that $s = s_0 c_1 \cdots s_{k-1} c_k s_k$ and $t = t_0 c_1 \cdots t_{k-1} c_k t_k$. Let $u = sau'$ and $v = tav'$. By construction of s and since $\mathrm{alph}(t) \subseteq \mathrm{alph}(s)$ we have

$$h(pt_0 a_1 \cdots t_{i-1} a_i) \mathrel{\mathcal{R}} h(pt_0 a_1 \cdots t_{i-1} a_i s_i)$$

for all $i \in \{0, \ldots, k\}$. Note that $h(a_{i+1} s_{i+1} \cdots a_k s_k u') \mathrel{\mathcal{L}} h(s_i a_{i+1} s_{i+1} \cdots a_k s_k u')$. Moreover, $h(p) >_{\mathcal{R}} h(pta)$ and thus $D_{\mathcal{R}}(p) \subsetneq D_{\mathcal{R}}(pta)$. Using the formulae $\langle\varphi\rangle_{>\mathsf{X}a}$ from Lemma 2.1 yields $u' \leqslant^{\mathrm{FO}^2}_{2,n-1} v'$. As $n - 1 \geqslant |D_{\mathcal{R}}(pv) \setminus D_{\mathcal{R}}(pta)| + |D_{\mathcal{L}}(u')| + 1$ holds, we can apply induction to obtain

$$\begin{aligned} u' &= s_{k+1} a_{k+2} \cdots s_{\ell-1} a_\ell s_\ell, \\ v' &= t_{k+1} a_{k+2} \cdots t_{\ell-1} a_\ell t_\ell. \end{aligned}$$

Setting $a_{k+1} = a$ yields the desired factorizations. The ranker property holds since all markers a_i are defined by $\mathcal{R}$- and $\mathcal{L}$-factorizations. □

Lemma 2.5. *Let $u = s_0 a_1 \cdots s_{\ell-1} a_\ell s_\ell$ and $v = t_0 a_1 \cdots t_{\ell-1} a_\ell t_\ell$ with $a_i \in A$, $s_i, t_i \in A^*$. Moreover, suppose for every $1 \leqslant i \leqslant \ell$ there exists a ranker r in*

$$\mathcal{Z} = \{\mathsf{X}_{b_1} \cdots \mathsf{X}_{b_k}, \mathsf{Y}_{b_k} \cdots \mathsf{Y}_{b_1} \mid b_1 \cdots b_k \text{ is a subword of } a_1 \cdots a_\ell \text{ for } b_i \in A\}$$

such that $r(u) = |s_0 a_1 \cdots s_{i-1} a_i|$ and $r(v) = |t_0 a_1 \cdots t_{i-1} a_i|$. If $u \leqslant^{\mathrm{FO}^2}_{m,n+\ell} v$ for $m \geqslant 2$ and $n \geqslant 0$, then $s_i \leqslant^{\mathrm{FO}^2}_{m-1,n} t_i$ for all $0 \leqslant i \leqslant \ell$.

Proof. For a ranker r let us say informally that it *reaches the occurrence* of the marker a_i if $r(u) = |s_0 a_1 \cdots s_{i-1} a_i|$ and $r(v) = |t_0 a_1 \cdots t_{i-1} a_i|$. Let $\mathcal{X}$ consist of the rankers in $\mathcal{Z}$ starting with an X-modality, and let $\mathcal{Y} = \mathcal{Z} \setminus \mathcal{X}$ be the rankers starting with a Y-modality. We perform an induction on the number of markers reachable by some ranker in $\mathcal{X}$. First, suppose all a_i are reached by a ranker in $\mathcal{Y}$. Using the formulae $\langle\varphi\rangle_{<\mathsf{Y}a_i}$ from Lemma 2.2 we get for every $1 \leqslant i < \ell$ that $s_0 a_1 s_1 \cdots a_i s_i \leqslant^{\mathrm{FO}^2}_{m,n+1} t_0 a_1 t_1 \cdots a_i t_i$ and $s_0 \leqslant^{\mathrm{FO}^2}_{m,n} t_0$. Hence using the formulae $\langle\varphi\rangle_{>\mathsf{Y}a_i}$ from Lemma 2.2, we see $s_i \leqslant^{\mathrm{FO}^2}_{m-1,n} t_i$ for all i.

Let now k be minimal such that the occurrence of a_k is reached by a ranker in $\mathcal{X}$. The same reasoning as above shows $s_i \leqslant^{\mathrm{FO}^2}_{m-1,n} t_i$ for all $i \leqslant k-2$. Using the formulae $\langle\varphi\rangle_{>\mathsf{X}a_k}$ from Lemma 2.1 we have

$$s_k a_{k+1} \cdots s_{\ell-1} a_\ell s_\ell \leqslant^{\mathrm{FO}^2}_{m,n+\ell-1} t_k a_{k+1} \cdots t_{\ell-1} a_\ell t_\ell$$

and induction yields $s_i \leqslant^{\mathrm{FO}^2}_{m-1,n} t_i$ for all $i \geqslant k$. It remains to show $s_{k-1} \leqslant^{\mathrm{FO}^2}_{m-1,n} t_{k-1}$. Let k' be the minimal index with $k' > k$ such that the occurrence of $a_{k'}$ is reached by a ranker in $\mathcal{Y}$. Then $s_0 a_1 s_1 \cdots a_{k'-1} s_{k'-1} \leqslant^{\mathrm{FO}^2}_{m,n+1} t_0 a_1 t_1 \cdots a_{k'-1} t_{k'-1}$. Using the formulae $\langle\varphi\rangle_{(\mathsf{Y}a_{k-1};\mathsf{X}a_k)}$ from Lemma 2.3 finally yields $s_{k-1} \leqslant^{\mathrm{FO}^2}_{m-1,n} t_{k-1}$. □

The preceding lemmas enable induction on the parameter m. We start with a homomorphism onto a monoid satisfying $U_m \leqslant V_m$ and want to show that preimages of $\leqslant$-order ideals are unions of $\leqslant^{\mathrm{FO}^2}_{m,n}$-order ideals for some sufficiently large n. Intuitively, a string rewriting technique yields the largest quotient which satisfies the inequality $U_{m-1} \leqslant V_{m-1}$. One rewriting step corresponds to one application of the inequality $U_{m-1} \leqslant V_{m-1}$ of level $m-1$. We then show that rewriting steps in appropriate contexts can be simulated by the inequality $U_m \leqslant V_m$.

Proposition 2.1. *Let $m \geqslant 1$ be an integer, let $h\colon A^* \to M$ be a surjective homomorphism onto a finite ordered monoid $(M, \leqslant)$ in* **DA** *which satisfies $U_m \leqslant V_m$. Then there exists a positive integer n such that $u \leqslant^{\mathrm{FO}^2}_{m,n} v$ implies $h(u) \leqslant h(v)$ for all $u, v \in A^*$.*

Proof. We perform an induction on m. For the base case $m = 1$ a result of Pin [19] shows that for every $\leqslant$-order ideal I of M the set $h^{-1}(I)$ is a finite union of languages $A^*a_1 \cdots A^*a_kA^*$ for some $k \geqslant 1$ and $a_i \in A$. Let n be the maximum of all indices k appearing in those unions when considering all order ideals $I \subseteq M$. If $u \leqslant^{\mathrm{FO}^2}_{1,n} v$, then for all languages $P = A^*a_1 \cdots A^*a_kA^*$ with $k \leqslant n$ we have that $v \in P$ implies $u \in P$. Moreover, the preimage L of the order ideal generated by $h(v)$ is a finite union of languages $A^*a_1 \cdots A^*a_kA^*$ with $k \leqslant n$. We have $v \in L$ and thus $u \in L$. This shows $h(u) \leqslant h(v)$.

In the following let $m \geqslant 2$ and fix some integer $\omega \geqslant 1$ such that x^ω is idempotent for all $x \in M$. We introduce a string rewriting system $\to$ on A^* by letting $t \to s$ if $h(s) = h(t)$ or if $t = pv_{m-1}q$ and $s = pu_{m-1}q$ for $p, q \in A^*$ and for $i \geqslant 2$ we have

$$v_1 = 1, \quad v_i = (u_{i-1}x_i)^\omega v_{i-1}(y_iu_{i-1})^\omega,$$
$$u_1 = z, \quad u_i = (u_{i-1}x_i)^\omega u_{i-1}(y_iu_{i-1})^\omega$$

for $x_i, y_i, z \in A^*$. Note that $t \to s$ implies $p'tq' \to p'sq'$ for all $p', q' \in A^*$. Let $\overset{*}{\to}$ be the transitive closure of $\to$, *i.e.*, let $t \overset{*}{\to} s$ if there exists a chain $t = w_1 \to w_2 \to \cdots \to w_\ell = s$ for some $\ell \geqslant 1$ and $w_i \in A^*$.

We now claim that, in certain contexts, we can lift the rewriting steps of $t \overset{*}{\to} s$ to M in an order respecting way.

Claim 1. *Let $u, v, s, t \in A^*$ with $t \overset{*}{\to} s$. If both $h(u) \mathrel{\mathcal{R}} h(us)$ and $h(v) \mathrel{\mathcal{L}} h(sv)$, then $h(usv) \leqslant h(utv)$.*

The proof of the claim is by induction on the length of a minimal $\to$-chain from t to s. The claim is trivial if $h(t) = h(s)$. Suppose $t \overset{*}{\to} t' \to s$ and $t' = pv_{m-1}q$ and $s = pu_{m-1}q$. Since $h(u) \mathrel{\mathcal{R}} h(us)$, there exists $x \in A^*$ such that $h(u) = h(usx)$; and since $h(v) \mathrel{\mathcal{L}} h(sv)$ there exists $y \in A^*$ such that $h(v) = h(ysv)$. Now $h(u) = h\big(u(pu_{m-1}qx)^\omega\big)$ and $h(v) = h\big((yu_{m-1}q)^\omega v\big)$. By letting $x_m = qxp$ and $y_m = qyp$, the inequality $U_m \leqslant V_m$ of M yields

$$\begin{aligned} h(usv) &= h\big(up(u_{m-1}x_m)^\omega u_{m-1}(y_mu_{m-1})^\omega qv\big) \\ &\leqslant h\big(up(u_{m-1}x_m)^\omega v_{m-1}(y_mu_{m-1})^\omega qv\big) = h(ut'v); \end{aligned}$$

to see this observe that $(pu_{m-1}qx)^\omega p = p(u_{m-1}qxp)^\omega = p(u_{m-1}x_m)^\omega$. By induction $h(ut'v) \leqslant h(utv)$ and thus $h(usv) \leqslant h(utv)$. Note $\mathrm{alph}(t') \subseteq \mathrm{alph}(s)$ and thus, $h(u) \mathrel{\mathcal{R}}$

$h(us)$ implies $h(u)\ \mathcal{R}\ h(ut')$, and symmetrically $h(v)\ \mathcal{L}\ h(sv)$ implies $h(v)\ \mathcal{L}\ h(t'v)$. This completes the proof of the claim.

Let $t \sim s$ if $t \overset{*}{\to} s$ and $s \overset{*}{\to} t$. This is a congruence on A^*, and $M' = (A^*/{\sim})$ becomes a monoid. Let $h' \colon A^* \to M'$ be the canonical homomorphism mapping $u \in A^*$ to its equivalence class modulo $\sim$. The preorder $\overset{*}{\to}$ on A^* induces a partial order on M' (which is also denoted by $\leqslant$ for conciseness) by letting $h'(u) \leqslant h'(v)$ if $v \overset{*}{\to} u$. Now, $(M', \leqslant)$ forms an ordered monoid. Moreover, M' is a (non-ordered) quotient of M and, in particular, M' is finite and in $\mathbf{DA}$, and x^ω is idempotent for all $x \in M'$.

By construction $(M', \leqslant)$ satisfies the identity $U_{m-1} \leqslant V_{m-1}$ and induction yields n such that $u \leqslant^{\mathrm{FO}^2}_{m-1,n} v$ implies $h'(u) \leqslant h'(v)$. We shall show that $u \leqslant^{\mathrm{FO}^2}_{m,N} v$ implies $h(u) \leqslant h(v)$ for all $u, v \in A^*$ and $N = n + 2|M|$. Let $u, v \in A^*$ with $u \leqslant^{\mathrm{FO}^2}_{m,N} v$. Consider the factorizations $u = s_0 a_1 \cdots s_{\ell-1} a_\ell s_\ell$ and $v = t_0 a_1 \cdots t_{\ell-1} a_\ell t_\ell$ given by Lemma 2.4; note that $\ell \leqslant 2|M| - 2$. Applying Lemma 2.5 to this yields for all i:

- $s_i \leqslant^{\mathrm{FO}^2}_{m-1,n} t_i$ and thus $t_i \overset{*}{\to} s_i$ by choice of n,
- $h(t_0 a_1 \cdots t_{i-1} a_i)\ \mathcal{R}\ h(t_0 a_1 \cdots t_{i-1} a_i s_i)$ and
- $h(a_{i+1} s_{i+1} \cdots a_\ell s_\ell)\ \mathcal{L}\ h(s_i a_{i+1} s_{i+1} \cdots a_\ell s_\ell)$.

Here for conciseness $t_0 a_1 \cdots t_{i-1} a_i$ is the empty word if $i = 0$, and so is $a_{i+1} s_{i+1} \cdots a_\ell s_\ell$ if $i = \ell$. To see the first property note that by choice of n, we have $h'(s_i) \leqslant h'(t_i)$ for all i, that is, $t_i \overset{*}{\to} s_i$. Applying Claim 1 repeatedly to substitute s_i with t_i for increasing $i \in \{0, \dots, \ell\}$ yields the following chain of inequalities:

$$\begin{aligned} h(u) &= h(s_0 a_1 s_1 \cdots s_{\ell-1} a_\ell s_\ell) \\ &\leqslant h(t_0 a_1 s_1 \cdots s_{\ell-1} a_\ell s_\ell) \\ &\ \ \vdots \\ &\leqslant h(t_0 a_1 t_1 \cdots s_{\ell-1} a_\ell s_\ell) \\ &\leqslant h(t_0 a_1 t_1 \cdots t_{\ell-1} a_\ell s_\ell) \\ &\leqslant h(t_0 a_1 t_1 \cdots t_{\ell-1} a_\ell t_\ell) = h(v). \end{aligned}$$

This concludes the proof. □

The following lemma roughly shows that U_m is $\leqslant^{\mathrm{WI}}_{m,n}$-smaller than V_m for suitable n. In combination with Theorem 1 this shows that the inequality $U_m \leqslant V_m$ holds for the ordered syntactic monoid of FO^2_m-definable languages. To formalize this, for an assignment $h \colon \Omega \to A^*$ let h_N be the extension of h to omega-terms given by $h_N(xy) = h_N(x) h_N(y)$ and $h_N(x^\omega) = h_N(x)^N$.

Lemma 2.6. *Let $m, n \geqslant 0$. For all mappings $h \colon \Omega \to A^*$, all $p, q \in A^*$ and all $N \geqslant n$ we have $p\, h_N(U_m)\, q \leqslant^{\mathrm{WI}}_{m,n} p\, h_N(V_m)\, q$.*

Proof. We start by giving a rough sketch of the proof. The idea is that the rankers involved in $\leqslant^{\mathrm{WI}}_{m,n}$ reach the difference of U_m and V_m in the center only with the very last direction alternation. Before that the rankers identify the same position of U_m and V_m (in the canonical sense). Moreover, since the center of U_m has a larger alphabet than that of V_m, a ranker which consists solely of X-modalities cannot pass the center of U_m

before it passes the center of V_m. Symmetrically, a ranker which only uses Y-modalities cannot pass the center in U_m before it passes it in V_m.

We now formalize these ideas. Let $N \geqslant n$ and let $U = ph_N(U_m)q$ and $V = ph_N(V_m)q$. We classify positions into environments of the center of increasing size; roughly speaking this defines a "two-dimensional" distance: the first dimension is how many direction alternations suffice to get near the center; the second is the depth sufficient for an one-way free ranker to decrease the direction alternation dimension. More precisely, for $2 \leqslant k \leqslant m$ and $n \leqslant N$ we let $i \in B_{k,n}$ if both

$$i > \left|p\, h_N\big((U_{m-1}x_m)^N \cdots (U_k x_{k+1})^N (U_{k-1}x_k)^{N-n}\big)\right|$$
$$|U| - i \geqslant \left|h_N\big((y_k U_{k-1})^{N-n}(y_{k+1}U_k)^N \cdots (y_m U_{m-1})^N\big)\, q\right|$$

hold and we let $B_1 = B_{1,n} = B_{2,0}$. Slightly abusing notation, we canonically identify positions of U with positions of V by removing the positions corresponding to the innermost occurrence of z in U_m. Specifically, if we let $\mathrm{pos}(U)$ and $\mathrm{pos}(V)$ be the set of positions of U and of V, respectively, then we identify $\mathrm{pos}(V)$ with the set $\mathrm{pos}(U) \setminus B_{1,N}$. By this convention, it also makes sense to interpret the set $B_{k,n}$ of position of U as a set of positions of V. Note that in this sense $\mathrm{pos}(V) \subseteq \mathrm{pos}(U)$ and $U[i] = V[i]$ for all $i \in \mathrm{pos}(V)$.

For $k \geqslant 1$ and $n \geqslant 0$ we show by induction on $k + n$ that for all $i, j \notin B_{k,n}$:

1. $R_{k,n}(V, i) \subseteq R_{k,n}(U, i)$ and
2. $r(V, i) < s(V, j) \Rightarrow r(U, i) < s(U, j)$ and $r(V, i) \leqslant s(V, j) \Rightarrow r(U, i) \leqslant s(U, j)$ for all rankers $r \in R_{k,n}(V, i)$ not ending with a Y-modality and all rankers $s \in R_{k,n}(V, j)$ not ending with an X-modality.

Here $R(w, i)$ consists of all rankers $r \in R$ such that $r(w, i)$ is defined (for a word w and a subset of rankers R). Choosing $k = m$ then implies the claim of the lemma.

The claim is vacuously true if $n = 0$. Suppose $n \geqslant 1$ in the following and let $i, j \notin B_{k,n}$. First consider $k = 1$. Condition (1) follows since all subwords of V of length at most n appearing to the left (respectively right) of i appear also as subwords of U to the left (respectively right) of i. (The converse may not be true, however.) Condition (2): For all $i \notin B_1$ and all $r \in \{\mathsf{X}_a \mid a \in A\}^+$ we have $r(U, i) \leqslant r(V, i)$. Symmetrically, for all $j \notin B_1$ and all $s \in \{\mathsf{Y}_a \mid a \in A\}^+$ we have $s(V, j) \leqslant s(U, j)$.

For the inductive step suppose $k \geqslant 2$. As a preparatory step we first consider a one-way ranker r with $|r| \leqslant n$, *i.e.*, a ranker $r \in \{\mathsf{X}_a \mid a \in A\}^+ \cup \{\mathsf{Y}_a \mid a \in A\}^+$ with $|r| \leqslant n$. We claim that $r(U, i) = r(V, i) \notin B_{k-1,N}$.

To see this first note that by symmetry it suffices to consider $i < \min(B_{k,n})$. If r only uses Y-modalities, then the claim is immediate. Let $r = \mathsf{X}_a r'$. Then $\mathsf{X}_a(V, i) < \min(B_{k,n-1})$ if and only if $\mathsf{X}_a(U, i) < \min(B_{k,n-1})$ and in this case $\mathsf{X}_a(V, i) = \mathsf{X}_a(U, i)$. Otherwise we already have $\mathsf{X}_a(V, i) > \max(B_{k-1,N})$ and moreover $\mathsf{X}_a(U, i) = \mathsf{X}_a(V, i)$. Note that the factor $h_N(U_{k-1}x_k)$ is in between i and $B_{k-1,M}$ and contains all labels of U and of V for positions in $B_{k-1,N}$. In both cases, induction yields the claim for r.

Condition (1): Let $r \in R_{k,n}(V, i)$. It suffices to consider $i < \min(B_{k,n})$. Let $r = r_0 r_1$ for a one-way ranker r_0 such that $r_1 \in R_{k-1,n} \cup \{\varepsilon\}$. The above yields $r_0(U, i) = r_0(V, i) \notin B_{k-1,N}$ and hence by induction $r(U, i) = r_1(U, r_0(U, i))$ is defined.

Condition (2): Let $r \in R_{k,n}(V, i)$ not end with a Y-modality and let $s \in R_{k,n}(V, j)$

not end with an X-modality. Assume $r(V,i) \lesssim s(V,j)$ for $\lesssim \in \{\leqslant, <\}$. Suppose $r = r_0 r_1$ and $s = s_0 s_1$ for one-way rankers r_0 and s_0 and $r_1, s_1 \in R_{k-1,n} \cup \{\varepsilon\}$. By the above, we have $r_0(V,i) = r_0(U,i) = i'$ and $s_0(V,j) = s_0(V,j) = j'$ and further $i', j' \notin B_{k-1,N}$. By assumption $r_1(V,i') \lesssim s_1(V,j')$; thus induction shows $r_1(U,i') \lesssim s_1(U,j')$. This implies $r(U,i) \lesssim s(U,j)$. □

We are now ready to prove our decidable characterization given in Theorem 2.

Theorem 2. For the direction from right to left consider the ordered syntactic monoid $(M, \leqslant)$, and let L be recognized by $h\colon A^* \to M$, *i.e.*, there exists a $\leqslant$-order ideal I such that $L = h^{-1}(I)$. We have $M \in \mathbf{DA}$ since L is FO^2 definable [27]. By Proposition 2.1 there exists n such that every preimage of a $\leqslant$-order ideal is a $\leqslant^{\mathrm{FO}^2}_{m,n}$-order ideal. In particular L is a $\leqslant^{\mathrm{FO}^2}_{m,n}$-order ideal and thus definable as the union of all $L(\varphi)$ over $\varphi \in \Sigma^2_{m,n}$ such that $L(\varphi) \subseteq L$. This union is finite because there are only finitely many languages definable in $\Sigma^2_{m,n}$.

For the direction from left to right suppose that L is definable in $\Sigma^2_{m,n}$ for some n. Lemma 2.6 shows $p\, h_N(U_m)\, q \leqslant^{\mathrm{WI}}_{m,n} p\, h_N(V_m)\, q$ for all assignments $h\colon \Omega \to A^*$, all $p, q \in A^*$ and all $N \geqslant n$. Theorem 1 yields $p\, h_N(U_m)\, q \leqslant^{\mathrm{FO}^2}_{m,n} p\, h_N(V_m)\, q$. In particular $p\, h_N(V_m)\, q \in L$ implies $p\, h_N(U_m)\, q \in L$. Choosing N such that all N^{th}-powers are idempotent in the syntactic monoid, this shows that the ordered syntactic monoid satisfies the inequality $U_m \leqslant V_m$. □

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