Reasoning about Shared Mutable Data Structure

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Basic Ideas

- Extention of Hoare logic for imperative programs
- Store/heap dichotomy
- Heap-manipulation commands (not expressions)
- Independent conjunction ($*$)
- Unrestricted address arithmetic
- Heap faults
- Nondeterministic allocation
- Inductive definitions over abstract structures

New Commands

$$
\begin{array}{l}
\langle \text{comm} \rangle ::= \cdots \\
\quad \mid \langle \text{var} \rangle := \mathbf{cons}(\langle \text{exp} \rangle, \ldots, \langle \text{exp} \rangle) \\
\quad \mid \langle \text{var} \rangle := [\langle \text{exp} \rangle] \\
\quad \mid [\langle \text{exp} \rangle] := \langle \text{exp} \rangle \\
\quad \mid \mathbf{dispose}\, \langle \text{exp} \rangle
\end{array}
$$

New Assertions

$$
\begin{array}{l}
\langle \text{assert} \rangle ::= \cdots \\
\quad \mid \langle \text{exp} \rangle \mapsto \langle \text{exp} \rangle \\
\quad \mid \mathbf{emptyheap} \\
\quad \mid \langle \text{assert} \rangle \ * \ \langle \text{assert} \rangle \\
\quad \mid \langle \text{assert} \rangle \mathrel{-\!\!*} \langle \text{assert} \rangle
\end{array}
$$

Why **cons**, **car**, and **cdr** do not occur in expressions

$$(\mathsf{x} = \mathsf{x})/\mathsf{x} \rightarrow \mathbf{cons}(1,2) \ \text{ is } \ \mathbf{cons}(1,2) = \mathbf{cons}(1,2)$$

$$(\mathsf{x} = \mathsf{x} \ * \ \mathsf{y} = \mathsf{y})/\mathsf{x} \rightarrow \mathbf{car}(\mathsf{z}), \mathsf{y} \rightarrow \mathbf{car}(\mathsf{z}) \ \text{ is}$$

$$(\mathbf{car}(\mathsf{z}) = \mathbf{car}(\mathsf{z})) \ * \ (\mathbf{car}(\mathsf{z}) = \mathbf{car}(\mathsf{z}))$$

Stores and Heaps

$$\begin{aligned}
\text{Atoms} &\subseteq \mathbf{Z} \\
\text{Locations} &\subseteq \mathbf{Z} \\
\text{Atoms} \cap \text{Locations} &= \{\} \\
\mathbf{nil} &\in \text{Atoms} \\
\\
\text{States} &= \text{Stores} \times \text{Heaps} \\
\text{Stores} &= \langle \text{var} \rangle \to \mathbf{Z} \\
\text{Heaps} &= \bigcup_{\substack{\text{fin} \\ A \subseteq \text{Locations}}} (A \to \mathbf{Z}) \\
\\
[\![-]\!]_{\text{exp}} &\in \langle \text{exp} \rangle \to \text{Stores} \to \mathbf{Z} \\
[\![-]\!]_{\text{boolexp}} &\in \langle \text{boolexp} \rangle \to \text{Stores} \to \mathbf{B} \\
[\![-]\!]_{\text{assert}} &\in \langle \text{assert} \rangle \to \text{States} \to \mathbf{B}
\end{aligned}$$

Structural Operational Semantics

An execution (of a command) is a sequence

$$\gamma_0 \rightsquigarrow \gamma_1 \rightsquigarrow \cdots$$

of *configurations* γ_i related by the *transition relation* $\rightsquigarrow$.

A configuration is either:

- *nonterminal*: A command-state pair.
- *terminal*: A state or **abort**.

We write:

- $\gamma \rightsquigarrow^* \gamma'$ iff there is a finite execution from γ to γ'.
- $\gamma \uparrow$ iff there is an infinite execution beginning with γ.

Our semantics is *total*:

- For every nonterminal configuration γ, there is at least one configuration γ' such that $\gamma \rightsquigarrow \gamma'$, i.e., executions do not get *stuck*.
- Every execution sequence can be extended to a sequence that is either infinite or ends with a terminal configuration.

The transition relation is defined by inference rules.

Transition Rules for the New Commands

- Construction

$$\frac{}{\langle v := \mathbf{cons}(e_0, \ldots, e_{n-1}), (s, h)\rangle \rightsquigarrow ([\, s \mid v{:}\, \ell \,], [\, h \mid \ell{:}\, [\![e_0]\!]_{\mathrm{exp}} s \mid \ldots \mid \ell + n - 1{:}\, [\![e_{n-1}]\!]_{\mathrm{exp}} s \,])}$$

  where $\ell, \ldots, \ell + n - 1 \in \mathrm{Locations} - \mathrm{dom}\, h$.

- Lookup

  When $[\![e]\!]_{\mathrm{exp}} s \in \mathrm{dom}\, h$:

$$\frac{}{\langle v := [e], (s, h)\rangle \rightsquigarrow ([\, s \mid v{:}\, h([\![e]\!]_{\mathrm{exp}} s) \,], h)}$$

  When $[\![e]\!]_{\mathrm{exp}} s \notin \mathrm{dom}\, h$:

$$\frac{}{\langle v := [e], (s, h)\rangle \rightsquigarrow \mathbf{abort}}$$

More Transition Rules

- Mutation

  When $[\![e]\!]_{\mathrm{exp}} s \in \mathrm{dom}\, h$:

$$\frac{}{\langle [e] := e', (s,h)\rangle \rightsquigarrow (s, [\, h \mid [\![e]\!]_{\mathrm{exp}} s \colon [\![e']\!]_{\mathrm{exp}} s \,])}$$

  When $[\![e]\!]_{\mathrm{exp}} s \notin \mathrm{dom}\, h$:

$$\frac{}{\langle [e] := e', (s,h)\rangle \rightsquigarrow \mathbf{abort}}$$

- Disposal

  When $[\![e]\!]_{\mathrm{exp}} s \in \mathrm{dom}\, h$:

$$\frac{}{\langle \mathbf{dispose}\ e, (s,h)\rangle \rightsquigarrow (s, h\rceil(\mathrm{dom}\, h - \{[\![e]\!]_{\mathrm{exp}} s\}))}$$

  When $[\![e]\!]_{\mathrm{exp}} s \notin \mathrm{dom}\, h$:

$$\frac{}{\langle \mathbf{dispose}\ e, (s,h)\rangle \rightsquigarrow \mathbf{abort}}$$

Semantics of the New Assertions

$$[\![e \mapsto e']\!]_{\text{assert}}(s,h)$$
$$\text{iff } \operatorname{dom} h = \{[\![e]\!]_{\text{exp}} s\} \text{ and } h([\![e]\!]_{\text{exp}} s) = [\![e']\!]_{\text{exp}} s,$$

$$[\![\mathbf{emptyheap}]\!]_{\text{assert}}(s,h)$$
$$\text{iff } \operatorname{dom} h = \{\},$$

$$[\![p_0 \ * \ p_1]\!]_{\text{assert}}(s,h)$$
$$\text{iff } \exists h_0, h_1.\ h_0 \perp h_1 \text{ and } h_0 \cdot h_1 = h$$
$$\text{and } [\![p_0]\!]_{\text{assert}}(s,h_0) \text{ and } [\![p_1]\!]_{\text{assert}}(s,h_1),$$

$$[\![p_0 \ -\!\!* \ p_1]\!]_{\text{assert}}(s,h)$$
$$\text{iff } \forall h'.\ (h' \perp h \text{ and } [\![p_0]\!]_{\text{assert}}(s,h'))$$
$$\text{implies } [\![p_1]\!]_{\text{assert}}(s, h \cdot h'),$$

where

$$h_0 \perp h_1 \text{ iff } \operatorname{dom} h_0 \cap \operatorname{dom} h_1 = \{\}$$

$$\operatorname{dom}(h_0 \cdot h_1) = \operatorname{dom} h_0 \cup \operatorname{dom} h_1$$

$$(h_0 \cdot h_1)\ell = \begin{cases} h_0\ell & \text{when } \ell \in \operatorname{dom} h_0 \\ h_1\ell & \text{when } \ell \in \operatorname{dom} h_1. \end{cases}$$

Pure Assertions

An assertion is *pure* if it does not contain $\mapsto$ or **emptyheap**.

A pure assertion does not depend upon the heap.

Pure Commands

A command is *pure* if it does not contain any of the new commands

$$v := \mathbf{cons}(e_0, \ldots, e_{n-1}) \qquad v := [e] \qquad [e] := e' \qquad \mathbf{dispose}\ e.$$

A pure command does not depend upon the heap and does not alter the heap.

Some Abbreviations

$$e \mapsto - \stackrel{\text{def}}{=} \exists x'.\, e \mapsto x' \quad \text{where } x' \text{ not free in } e$$

$$e \hookrightarrow e' \stackrel{\text{def}}{=} e \mapsto e' * \mathbf{true}$$

$$e \mapsto e_0, \ldots, e_{n-1} \stackrel{\text{def}}{=} e \mapsto e_0 * \cdots * e + n - 1 \mapsto e_{n-1}$$

$$e \hookrightarrow e_0, \ldots, e_{n-1} \stackrel{\text{def}}{=} e \hookrightarrow e_0 * \cdots * e + n - 1 \hookrightarrow e_{n-1}$$

$$\text{iff } e \mapsto e_0, \ldots, e_{n-1} * \mathbf{true}$$

An Example of $*$

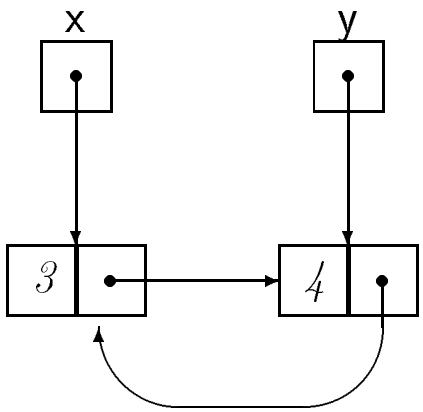
$$[\![p_0 * p_1]\!]_{\text{assert}}(s,h)$$

$$\text{iff } \exists h_0, h_1.\ h_0 \perp h_1 \text{ and } h_0 \cdot h_1 = h$$

$$\text{and } [\![p_0]\!]_{\text{assert}}(s,h_0) \text{ and } [\![p_1]\!]_{\text{assert}}(s,h_1)$$

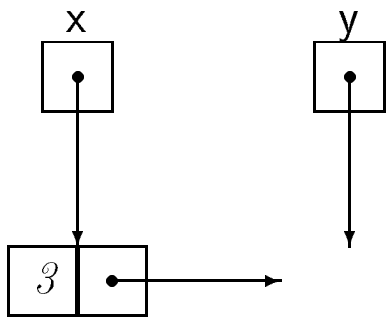
$\mathsf{x} \mapsto 3, \mathsf{y} * \mathsf{y} \mapsto 4, \mathsf{x}$

$\mathsf{x} \hookrightarrow 3, \mathsf{y} * \mathsf{y} \hookrightarrow 4, \mathsf{x}$



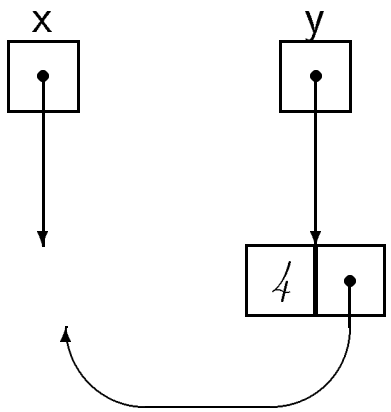
$\mathsf{x} \mapsto 3, \mathsf{y}$

$\mathsf{x} \hookrightarrow 3, \mathsf{y}$



$\mathsf{y} \mapsto 4, \mathsf{x}$

$\mathsf{y} \hookrightarrow 4, \mathsf{x}$



More Examples of $*$

Suppose $s\,\mathsf{x}$ and $s\,\mathsf{y}$ are distinct locations, so that

$$h_1 = \{\langle s\,\mathsf{x}, \langle 1 \rangle\rangle\} \qquad \text{and} \qquad h_2 = \{\langle s\,\mathsf{y}, \langle 2 \rangle\rangle\}.$$

are heaps with disjoint domains. Then

If P is:	then $[\![P]\!](s,h)$ is:
$\mathsf{x} \mapsto 1$	$h = h_1$
$\mathsf{y} \mapsto 2$	$h = h_2$
$\mathsf{x} \mapsto 1 \ * \ \mathsf{y} \mapsto 2$	$h = h_1 \cdot h_2$
$\mathsf{x} \mapsto 1 \ * \ \mathsf{x} \mapsto 1$	**false**
$\mathsf{x} \mapsto 1 \vee \mathsf{y} \mapsto 2$	$h = h_1$ or $h = h_2$
$\mathsf{x} \mapsto 1 \ * \ (\mathsf{x} \mapsto 1 \vee \mathsf{y} \mapsto 2)$	$h = h_1 \cdot h_2$
$(\mathsf{x} \mapsto 1 \vee \mathsf{y} \mapsto 2) \ * \ (\mathsf{x} \mapsto 1 \vee \mathsf{y} \mapsto 2)$	$h = h_1 \cdot h_2$
$\mathsf{x} \mapsto 1 \ * \ \mathsf{y} \mapsto 2 \ * \ (\mathsf{x} \mapsto 1 \vee \mathsf{y} \mapsto 2)$	**false**
$\mathsf{x} \mapsto 1 \ * \ \mathbf{true}$	$h_1 \subseteq h$
$\mathsf{x} \mapsto 1 \ * \ \neg\, \mathsf{x} \mapsto 1$	$h_1 \subseteq h$

Rules and Axiom Schemata for $*$ and $-\!\!*$

$$p_1 * p_2 \Leftrightarrow p_2 * p_1$$

$$(p_1 * p_2) * p_3 \Leftrightarrow p_1 * (p_2 * p_3)$$

$$p * \mathbf{emptyheap} \Leftrightarrow p$$

$$(p_1 \vee p_2) * q \Leftrightarrow (p_1 * q) \vee (p_2 * q)$$

$$(p_1 \wedge p_2) * q \Rightarrow (p_1 * q) \wedge (p_2 * q)$$

$$(\exists x.\, p_1) * p_2 \Leftrightarrow \exists x.\, (p_1 * p_2) \quad \text{when } x \text{ not free in } p_2$$

$$(\forall x.\, p_1) * p_2 \Rightarrow \forall x.\, (p_1 * p_2) \quad \text{when } x \text{ not free in } p_2.$$

$$\frac{p_1 \Rightarrow p_2 \qquad q_1 \Rightarrow q_2}{p_1 * q_1 \Rightarrow p_2 * q_2.}$$

$$\frac{p_1 * p_2 \Rightarrow p_3}{p_1 \Rightarrow (p_2 -\!\!* p_3)} \qquad \frac{p_1 \Rightarrow (p_2 -\!\!* p_3)}{p_1 * p_2 \Rightarrow p_3}$$

Axiom Schemata for $\mapsto$

$$x \mapsto e \wedge x' \mapsto e' \Leftrightarrow x \mapsto e \wedge x = x' \wedge e = e'$$

$$(x \mapsto e * \mathbf{true}) \wedge \mathbf{emptyheap} \Leftrightarrow \mathbf{false}$$

$$x \mapsto e * x' \mapsto e' \Rightarrow x \neq x'.$$

Axiom Schemata for Purity

$$p_1 \wedge p_2 \Rightarrow p_1 \ * \ p_2 \quad \text{when } p_1 \text{ or } p_2 \text{ is pure}$$

$$p_1 \ * \ p_2 \Rightarrow p_1 \wedge p_2 \quad \text{when } p_1 \text{ and } p_2 \text{ are pure}$$

$$(p \wedge q) \ * \ r \Leftrightarrow (p \ * \ r) \wedge q \qquad \text{when } q \text{ is pure}$$

$$(p_1 \mathrel{-\!\!*} p_2) \Rightarrow (p_1 \Rightarrow p_2) \quad \text{when } p_1 \text{ is pure}$$

$$(p_1 \Rightarrow p_2) \Rightarrow (p_1 \mathrel{-\!\!*} p_2) \quad \text{when } p_1 \text{ and } p_2 \text{ are pure}$$

In addition, the model validates all of the laws of classical logic for $\neg$, $\Rightarrow$, $\wedge$ and $\vee$. It uses the boolean algebra structure of the powerset of Heaps, which amounts to pointwise interpretations for each classical connective.

Two Unsound Axiom Schemata

$$p \not\Rightarrow p * p \qquad \text{(Contraction)}$$

e.g. $p : \mathsf{x} \mapsto 1$

$$p * q \not\Rightarrow p \qquad \text{(Weakening)}$$

e.g. $p : \mathsf{x} \mapsto 1$

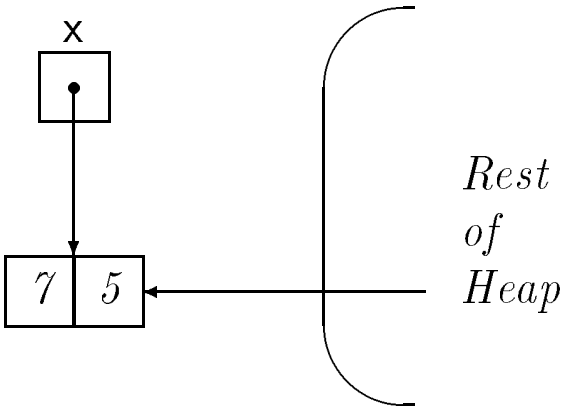
$q : \mathsf{y} \mapsto 2$

An Example of $-\!\!*$

$[\![p_0 -\!\!* p]\!]_{\text{assert}}$ iff $\forall h'.\ (h' \perp h$ and $[\![p_0]\!]_{\text{assert}}(s, h'))$
implies $[\![p_1]\!]_{\text{assert}}(s, h \cdot h')$

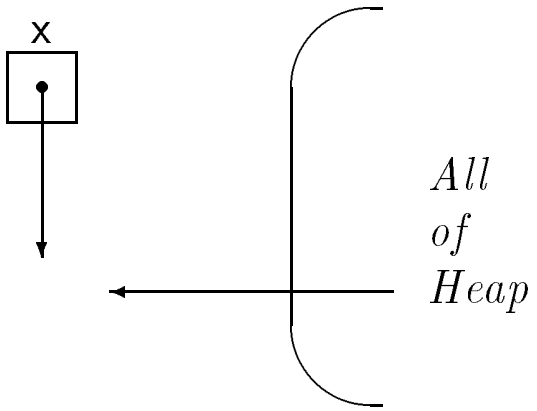
If:

P



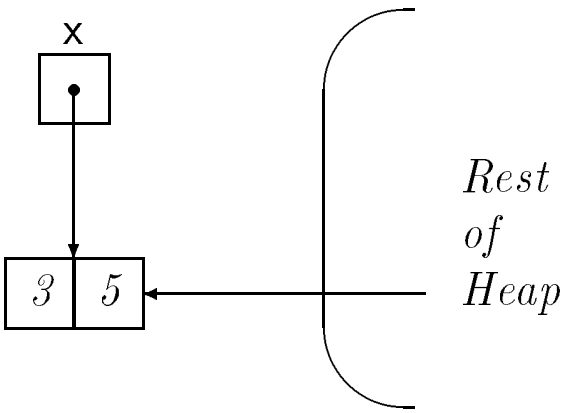
then:

$\mathsf{x} \mapsto 7, 5 -\!\!* P$



and:

$\mathsf{x} \mapsto 3, 5 \ * \ (\mathsf{x} \mapsto 7, 5 -\!\!* P)$



Specifications

$$[\![-]\!]_{\text{spec}} \in \langle \text{spec} \rangle \to \mathbf{B}.$$

Partial correctness:

$$[\![\{p\}\ c\ \{q\}]\!]_{\text{spec}} \text{ iff } \forall \sigma \in \text{States}.\ [\![p]\!]_{\text{assert}}\sigma \text{ implies}$$
$$\neg\ c, \sigma \rightsquigarrow^* \mathbf{abort}$$
$$\text{and } (\forall \sigma'.\ c, \sigma \rightsquigarrow^* \sigma' \text{ implies } [\![q]\!]_{\text{assert}}\sigma')$$

Total correctness:

$$[\![[\ p\ ]\ c\ [\ q\ ]]\!]_{\text{spec}} \text{ iff } \forall \sigma \in \text{States}.\ [\![p]\!]_{\text{assert}}\sigma \text{ implies}$$
$$\neg\ c, \sigma \rightsquigarrow^* \mathbf{abort}$$
$$\text{and } \neg\ c, \sigma \uparrow$$
$$\text{and } (\forall \sigma'.\ c, \sigma \rightsquigarrow^* \sigma' \text{ implies } [\![q]\!]_{\text{assert}}\sigma').$$

- Specifications are implicitly quantified over both stores and heaps — and over all executions.
- An execution giving a memory fault falsifies a specification.

An Inference Rule for Mutation

$$\begin{array}{l}\{p \ * \ (e \mapsto -)\} \\ [e] := e' \\ \{p \ * \ (e \mapsto e')\}\end{array}$$

An Inference Rule for Construction

$$\begin{array}{l}\{p\} \\ v := \mathbf{cons}(e_0, \ldots, e_{n-1}) \\ \{p \ * \ (v \mapsto e_0, \ldots, e_{n-1})\}\end{array}$$

where v is not free in e_0, ..., e_{n-1}, or p.

A Stronger Rule

$$\begin{array}{l}\{p\} \\ v := \mathbf{cons}(e_0, \ldots, e_{n-1}) \\ \{\exists v'.\ p' \ * \ (v \mapsto e'_0, \ldots, e'_{n-1})\}\end{array}$$

where

- v' is distinct from v and not free in e_0, ..., e_{n-1}, or p,
- p' is the substitution $p/v \to v'$,
- each e'_i is the subsitution $e_i/v \to v'$.

Violating Record Boundaries

$$
\begin{array}{l}
\{p\} \\
\mathsf{x} := \mathbf{cons}(1,2) \\
\qquad \vdots \qquad\qquad \text{(no assignment to } \mathsf{x}\text{)} \\
[\mathsf{x}+2] := 3 \\
\{q\}
\end{array}
$$

cannot hold, since there will be some execution in which $s\,\mathsf{x}+2 \notin \operatorname{dom} h$ before the execution of $[\mathsf{x}+2] := 3$. If one tries to prove such a specification:

$$
\begin{array}{lll}
\{p\} & & \\
\mathsf{x} := \mathbf{cons}(1,2) & & \\
\{\exists \mathsf{x}'.\ (p/\mathsf{x} \to \mathsf{x}') \ *\ \mathsf{x} \mapsto 1,2\} & \equiv \{\exists \mathsf{x}'.\ (p/\mathsf{x} \to \mathsf{x}') & \\
 & \quad *\ \mathsf{x} \mapsto 1 \ *\ \mathsf{x}+1 \mapsto 2\} & \\
\qquad \vdots & \qquad \vdots \text{ (no implication)} & \\
\{q \ *\ (\mathsf{x}+2 \mapsto -)\} & \equiv \{q \ *\ (\exists \mathsf{y}.\ \mathsf{x}+2 \mapsto \mathsf{y})\} & \\
[\mathsf{x}+2] := 3 & & \\
\{q \ *\ (\mathsf{x}+2 \mapsto 3)\} & &
\end{array}
$$

An Inference Rule for Lookup

$$\{\exists v'.\ p' \wedge e \hookrightarrow v'\}$$
$$v := [e]$$
$$\{p\}$$

where v' does not occur free in p or e, and p' is $p/v \rightarrow v'$.

Disposal

$$\{p\ *\ (e \mapsto -)\}$$
$$\textbf{dispose}\ e$$
$$\{p\}$$

Frame Introduction (Generalized Constancy)

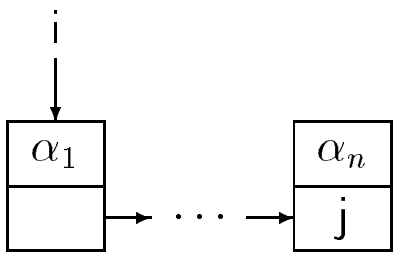
$$\frac{\{p\}\ c\ \{q\}}{\{p\ *\ r\}\ c\ \{q\ *\ r\}}$$

when no variable occurs both free in r and free on the left of an assignment command in c.

For all of the new inference rules, there are similar rules for total correctness.

An Inductive Definition: Singly-Linked Lists

$\mathsf{list}\ \alpha\ (\mathsf{i},\mathsf{j})$:



Definition

$$\mathsf{list}\ \epsilon\ (\mathsf{i},\mathsf{j}) \stackrel{\text{def}}{=} \mathbf{emptyheap} \wedge \mathsf{i} = \mathsf{j}$$

$$\mathsf{list}\ \mathsf{a}{\cdot}\alpha\ (\mathsf{i},\mathsf{k}) \stackrel{\text{def}}{=} \exists \mathsf{j}.\ \mathsf{i} \mapsto \mathsf{a},\mathsf{j}\ *\ \mathsf{list}\ \alpha\ (\mathsf{j},\mathsf{k})$$

Properties

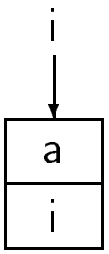
$$\mathsf{list}\ \mathsf{a}\ (\mathsf{i},\mathsf{j}) \Leftrightarrow \mathsf{i} \mapsto \mathsf{a},\mathsf{j}$$

$$\mathsf{list}\ \alpha{\cdot}\beta\ (\mathsf{i},\mathsf{k}) \Leftrightarrow \exists \mathsf{j}.\ \mathsf{list}\ \alpha\ (\mathsf{i},\mathsf{j})\ *\ \mathsf{list}\ \beta\ (\mathsf{j},\mathsf{k})$$

$$\mathsf{list}\ \alpha{\cdot}\mathsf{b}\ (\mathsf{i},\mathsf{k}) \Leftrightarrow \exists \mathsf{j}.\ \mathsf{list}\ \alpha\ (\mathsf{i},\mathsf{j})\ *\ \mathsf{j} \mapsto \mathsf{b},\mathsf{k}$$

Cycles

When



we have

- $\mathsf{list}\ \epsilon\ (\mathsf{i},\mathsf{i})$ holds for the empty heap.
- $\mathsf{list}\ \mathsf{a}\ (\mathsf{i},\mathsf{i})$ holds for the two-element heap pictured above.

But because of the $*$ in

$$\mathsf{list}\ \mathsf{a}{\cdot}\alpha\ (\mathsf{i},\mathsf{k}) \stackrel{\text{def}}{=} \exists \mathsf{j}.\ \mathsf{i} \mapsto \mathsf{a},\mathsf{j}\ *\ \mathsf{list}\ \alpha\ (\mathsf{j},\mathsf{k}),$$

$\mathsf{list}\ \mathsf{a}{\cdot}\mathsf{a}\ (\mathsf{i},\mathsf{i})$, $\mathsf{list}\ \mathsf{a}{\cdot}\mathsf{a}{\cdot}\mathsf{a}\ (\mathsf{i},\mathsf{i})$, ... do not hold.

Emptyness Conditions

$$\mathsf{list}\ \alpha\ (\mathsf{i},\mathsf{j}) \Rightarrow \left(\begin{array}{ll} & (\mathsf{i} = \mathbf{nil} \Rightarrow (\alpha = \epsilon \wedge \mathsf{j} = \mathbf{nil})) \\ \wedge & (\mathsf{i} \neq \mathsf{j} \Rightarrow \alpha \neq \epsilon). \end{array}\right)$$

Inserting a List Element

- At the beginning:

$$
\begin{array}{l}
\{\mathbf{list}\ \alpha\,(\mathsf{i},\mathsf{j})\} \\
\mathsf{k} := \mathbf{cons}(\mathsf{a},\mathsf{i}); \\
\{\mathbf{list}\ \alpha\,(\mathsf{i},\mathsf{j})\ *\ \mathsf{k} \mapsto \mathsf{a},\mathsf{i}\} \\
\{\exists \mathsf{i}.\ \mathsf{k} \mapsto \mathsf{a},\mathsf{i}\ *\ \mathbf{list}\ \alpha\,(\mathsf{i},\mathsf{j})\} \\
\{\mathbf{list}\ \mathsf{a}{\cdot}\alpha\,(\mathsf{k},\mathsf{j})\} \\
\mathsf{i} := \mathsf{k} \\
\{\mathbf{list}\ \mathsf{a}{\cdot}\alpha\,(\mathsf{i},\mathsf{j})\}
\end{array}
$$

- At the end of a nonempty segment:

$$
\begin{array}{l}
\{\mathbf{list}\ \alpha\,(\mathsf{i},\mathsf{j})\ *\ \mathsf{j} \mapsto \mathsf{a},\mathsf{k}\} \\
\mathsf{l} := \mathbf{cons}(\mathsf{b},\mathsf{k})\ ; \\
\{\mathbf{list}\ \alpha\,(\mathsf{i},\mathsf{j})\ *\ \mathsf{j} \mapsto \mathsf{a},\mathsf{k}\ *\ \mathsf{l} \mapsto \mathsf{b},\mathsf{k}\} \\
\{\mathbf{list}\ \alpha\,(\mathsf{i},\mathsf{j})\ *\ \mathsf{j} \mapsto \mathsf{a}\ *\ \mathsf{j}+1 \mapsto \mathsf{k}\ *\ \mathsf{l} \mapsto \mathsf{b},\mathsf{k}\} \\
\{\mathbf{list}\ \alpha\,(\mathsf{i},\mathsf{j})\ *\ \mathsf{j} \mapsto \mathsf{a}\ *\ \mathsf{j}+1 \mapsto -\ *\ \mathsf{l} \mapsto \mathsf{b},\mathsf{k}\} \\
[\mathsf{j}+1] := \mathsf{l} \\
\{\mathbf{list}\ \alpha\,(\mathsf{i},\mathsf{j})\ *\ \mathsf{j} \mapsto \mathsf{a}\ *\ \mathsf{j}+1 \mapsto \mathsf{l}\ *\ \mathsf{l} \mapsto \mathsf{b},\mathsf{k}\} \\
\{\mathbf{list}\ \alpha\,(\mathsf{i},\mathsf{j})\ *\ \mathsf{j} \mapsto \mathsf{a},\mathsf{l}\ *\ \mathsf{l} \mapsto \mathsf{b},\mathsf{k}\} \\
\{\mathbf{list}\ \alpha{\cdot}\mathsf{a}\,(\mathsf{i},\mathsf{l})\ *\ \mathsf{l} \mapsto \mathsf{b},\mathsf{k}\} \\
\{\mathbf{list}\ \alpha{\cdot}\mathsf{a}{\cdot}\mathsf{b}\,(\mathsf{i},\mathsf{k})\}
\end{array}
$$

Deleting a List Element

- At the beginning:

$$\{\mathbf{list}\ \mathsf{a}{\cdot}\alpha\,(\mathsf{i},\mathsf{k})\}$$
$$\{\exists \mathsf{j}.\ \mathsf{i}\mapsto \mathsf{a},\mathsf{j}\ *\ \mathbf{list}\ \alpha\,(\mathsf{j},\mathsf{k})\}$$
$$\vdots$$
$$\{\exists \mathsf{j}.\ (\mathsf{i}\mapsto \mathsf{a},\mathsf{j}\ *\ \mathbf{list}\ \alpha\,(\mathsf{j},\mathsf{k}))\wedge \mathsf{i}+1\hookrightarrow \mathsf{j}\}$$
$$\mathsf{j}:=[\mathsf{i}+1]\,;$$
$$\{\mathsf{i}\mapsto \mathsf{a},\mathsf{j}\ *\ \mathbf{list}\ \alpha\,(\mathsf{j},\mathsf{k})\}$$
$$\{\mathsf{i}\mapsto \mathsf{a}\ *\ \mathsf{i}+1\mapsto \mathsf{j}\ *\ \mathbf{list}\ \alpha\,(\mathsf{j},\mathsf{k})\}$$
$$\mathbf{dispose}\ \mathsf{i}\,;$$
$$\{\mathsf{i}+1\mapsto \mathsf{j}\ *\ \mathbf{list}\ \alpha\,(\mathsf{j},\mathsf{k})\}$$
$$\mathbf{dispose}\ \mathsf{i}+1\,;$$
$$\{\mathbf{list}\ \alpha\,(\mathsf{j},\mathsf{k})\}$$
$$\mathsf{i}:=\mathsf{j}$$
$$\{\mathbf{list}\ \alpha\,(\mathsf{i},\mathsf{k})\}$$

- At the end of a nonempty segment:

$$\{\mathbf{list}\ \alpha\,(\mathsf{i},\mathsf{j})\ *\ \mathsf{j}\mapsto \mathsf{a},\mathsf{k}\ *\ \mathsf{k}\mapsto \mathsf{b},\mathsf{l}\}$$
$$[\mathsf{j}+1]:=\mathsf{l}\,;$$
$$\{\mathbf{list}\ \alpha\,(\mathsf{i},\mathsf{j})\ *\ \mathsf{j}\mapsto \mathsf{a},\mathsf{l}\ *\ \mathsf{k}\mapsto \mathsf{b},\mathsf{l}\}$$
$$\mathbf{dispose}\ \mathsf{k}\,;\mathbf{dispose}\ \mathsf{k}+1$$
$$\{\mathbf{list}\ \alpha\,(\mathsf{i},\mathsf{j})\ *\ \mathsf{j}\mapsto \mathsf{a},\mathsf{l}\}$$
$$\{\mathbf{list}\ \alpha{\cdot}\mathsf{a}\,(\mathsf{i},\mathsf{l})\}$$

Reversing a List

$\{\mathbf{list}\ \gamma\,(\mathsf{i}, \mathbf{nil})\}$

$\{\mathbf{list}\ \gamma\,(\mathsf{i}, \mathbf{nil})\ *\ \mathbf{emptyheap}\}$

$\mathsf{j} := \mathbf{nil}\,;$

$\{\mathbf{list}\ \gamma\,(\mathsf{i}, \mathbf{nil})\ *\ (\mathbf{emptyheap} \wedge \mathsf{j} = \mathbf{nil})\}$

$\{\mathbf{list}\ \gamma\,(\mathsf{i}, \mathbf{nil})\ *\ \mathbf{list}\ \epsilon\,(\mathsf{j}, \mathbf{nil})\}$

$\{\exists \alpha, \beta.\ (\mathbf{list}\ \alpha\,(\mathsf{i}, \mathbf{nil})\ *\ \mathbf{list}\ \beta\,(\mathsf{j}, \mathbf{nil})) \wedge \gamma^\dagger = \alpha^\dagger \cdot \beta\}$

$\mathbf{while}\ \mathsf{i} \neq \mathbf{nil}\ \mathbf{do}$

$\big(\{\exists \mathsf{a}, \alpha, \beta.\ (\mathbf{list}\ \mathsf{a}\cdot\alpha\,(\mathsf{i}, \mathbf{nil})\ *\ \mathbf{list}\ \beta\,(\mathsf{j}, \mathbf{nil})) \wedge \gamma^\dagger = (\mathsf{a}\cdot\alpha)^\dagger \cdot \beta\}$

$\{\exists \mathsf{a}, \alpha, \beta, \mathsf{k}.\ (\mathsf{i} \mapsto \mathsf{a}, \mathsf{k}\ *\ \mathbf{list}\ \alpha\,(\mathsf{k}, \mathbf{nil})\ *\ \mathbf{list}\ \beta\,(\mathsf{j}, \mathbf{nil}))$
$\wedge\ \gamma^\dagger = (\mathsf{a}\cdot\alpha)^\dagger \cdot \beta\}$

$\mathsf{k} := [\mathsf{i} + 1]\,;$

$\{\exists \mathsf{a}, \alpha, \beta.\ (\mathsf{i} \mapsto \mathsf{a}, \mathsf{k}\ *\ \mathbf{list}\ \alpha\,(\mathsf{k}, \mathbf{nil})\ *\ \mathbf{list}\ \beta\,(\mathsf{j}, \mathbf{nil}))$
$\wedge\ \gamma^\dagger = (\mathsf{a}\cdot\alpha)^\dagger \cdot \beta\}$

$[\mathsf{i} + 1] := \mathsf{j}\,;$

$\{\exists \mathsf{a}, \alpha, \beta.\ (\mathsf{i} \mapsto \mathsf{a}, \mathsf{j}\ *\ \mathbf{list}\ \alpha\,(\mathsf{k}, \mathbf{nil})\ *\ \mathbf{list}\ \beta\,(\mathsf{j}, \mathbf{nil}))$
$\wedge\ \gamma^\dagger = (\mathsf{a}\cdot\alpha)^\dagger \cdot \beta\}$

$\{\exists \mathsf{a}, \alpha, \beta.\ (\mathbf{list}\ \alpha\,(\mathsf{k}, \mathbf{nil})\ *\ \mathbf{list}\ \mathsf{a}\cdot\beta\,(\mathsf{i}, \mathbf{nil})) \wedge \gamma^\dagger = \alpha^\dagger \cdot \mathsf{a} \cdot \beta\}$

$\{\exists \alpha, \beta.\ (\mathbf{list}\ \alpha\,(\mathsf{k}, \mathbf{nil})\ *\ \mathbf{list}\ \beta\,(\mathsf{i}, \mathbf{nil})) \wedge \gamma^\dagger = \alpha^\dagger \cdot \beta\}$

$\mathsf{j} := \mathsf{i}\,;\ \mathsf{i} := \mathsf{k}$

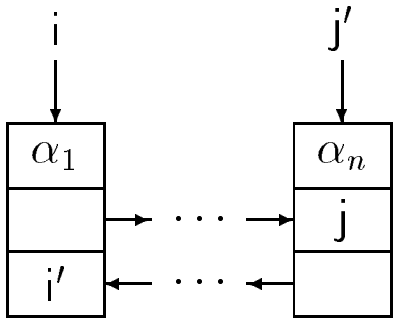
$\{\exists \alpha, \beta.\ (\mathbf{list}\ \alpha\,(\mathsf{i}, \mathbf{nil})\ *\ \mathbf{list}\ \beta\,(\mathsf{j}, \mathbf{nil})) \wedge \gamma^\dagger = \alpha^\dagger \cdot \beta\}\big)$

$\{\exists \alpha, \beta.\ \mathbf{list}\ \beta\,(\mathsf{j}, \mathbf{nil}) \wedge \gamma^\dagger = \alpha^\dagger \cdot \beta \wedge \alpha = \epsilon\}$

$\{\mathbf{list}\ \gamma^\dagger\,(\mathsf{j}, \mathbf{nil})\}$

An Inductive Definition: Doubly-Linked Lists

$\mathsf{dlist}\,\alpha\,(\mathsf{i},\mathsf{i}',\mathsf{j},\mathsf{j}')$:



Definition

$$\mathsf{dlist}\,\epsilon\,(\mathsf{i},\mathsf{i}',\mathsf{j},\mathsf{j}') \stackrel{\text{def}}{=} \mathbf{emptyheap} \wedge \mathsf{i} = \mathsf{j} \wedge \mathsf{i}' = \mathsf{j}'$$

$$\mathsf{dlist}\,\mathsf{a}{\cdot}\alpha\,(\mathsf{i},\mathsf{i}',\mathsf{k},\mathsf{k}') \stackrel{\text{def}}{=} \exists \mathsf{j}.\ \mathsf{i} \mapsto \mathsf{a},\mathsf{j},\mathsf{i}' \ * \ \mathsf{dlist}\,\alpha\,(\mathsf{j},\mathsf{i},\mathsf{k},\mathsf{k}').$$

Properties

$$\mathsf{dlist}\,\mathsf{a}\,(\mathsf{i},\mathsf{i}',\mathsf{j},\mathsf{j}') \Leftrightarrow \mathsf{i} \mapsto \mathsf{a},\mathsf{j},\mathsf{i}' \wedge \mathsf{i} = \mathsf{j}'$$

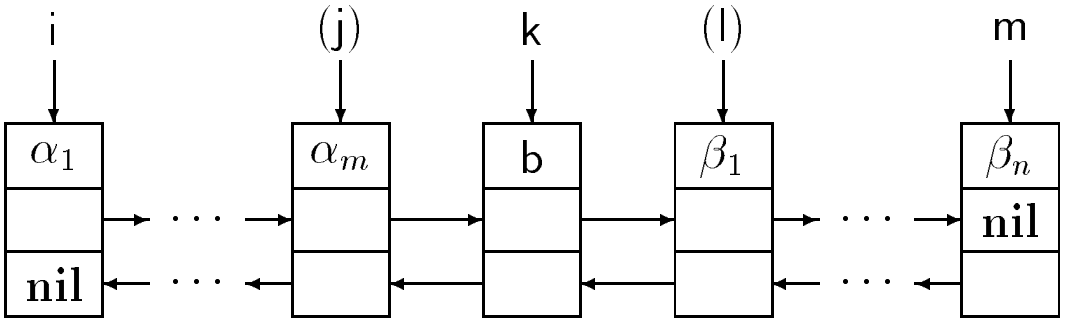
$$\mathsf{dlist}\,\alpha{\cdot}\beta\,(\mathsf{i},\mathsf{i}',\mathsf{k},\mathsf{k}') \Leftrightarrow \exists \mathsf{j},\mathsf{j}'.\ \mathsf{dlist}\,\alpha\,(\mathsf{i},\mathsf{i}',\mathsf{j},\mathsf{j}') \ * \ \mathsf{dlist}\,\beta\,(\mathsf{j},\mathsf{j}',\mathsf{k},\mathsf{k}')$$

$$\mathsf{dlist}\,\alpha{\cdot}\mathsf{b}\,(\mathsf{i},\mathsf{i}',\mathsf{k},\mathsf{k}') \Leftrightarrow \exists \mathsf{j}'.\ \mathsf{dlist}\,\alpha\,(\mathsf{i},\mathsf{i}',\mathsf{k}',\mathsf{j}') \ * \ \mathsf{k}' \mapsto \mathsf{b},\mathsf{k},\mathsf{j}'.$$

Emptyness Conditions

$$\mathsf{dlist}\,\alpha\,(\mathsf{i},\mathsf{i}',\mathsf{j},\mathsf{j}') \Rightarrow \left(\begin{array}{rl} & (\mathsf{i} = \mathbf{nil} \Rightarrow (\alpha = \epsilon \wedge \mathsf{j} = \mathbf{nil} \wedge \mathsf{i}' = \mathsf{j}')) \\ \wedge & (\mathsf{j}' = \mathbf{nil} \Rightarrow (\alpha = \epsilon \wedge \mathsf{i}' = \mathbf{nil} \wedge \mathsf{i} = \mathsf{j})) \\ \wedge & (\mathsf{i} \neq \mathsf{j} \Rightarrow \alpha \neq \epsilon) \\ \wedge & (\mathsf{i}' \neq \mathsf{j}' \Rightarrow \alpha \neq \epsilon) \end{array}\right)$$

Deleting an Element from a Doubly-Linked List



$\{\exists \mathsf{j}, \mathsf{l}.\ \mathsf{dlist}\,\alpha\,(\mathsf{i}, \mathbf{nil}, \mathsf{k}, \mathsf{j})\ *\ \mathsf{k} \mapsto \mathsf{b}, \mathsf{l}, \mathsf{j}\ *\ \mathsf{dlist}\,\beta\,(\mathsf{l}, \mathsf{k}, \mathbf{nil}, \mathsf{m})\}$

$\mathsf{l} := [\mathsf{k}+1]\ ;\ \mathsf{j} := [\mathsf{k}+2]\ ;$

$\{\mathsf{dlist}\,\alpha\,(\mathsf{i}, \mathbf{nil}, \mathsf{k}, \mathsf{j})\ *\ \mathsf{k} \mapsto \mathsf{b}, \mathsf{l}, \mathsf{j}\ *\ \mathsf{dlist}\,\beta\,(\mathsf{l}, \mathsf{k}, \mathbf{nil}, \mathsf{m})\}$

dispose k ; **dispose** $\mathsf{k}+1$; **dispose** $\mathsf{k}+2$;

$\{\mathsf{dlist}\,\alpha\,(\mathsf{i}, \mathbf{nil}, \mathsf{k}, \mathsf{j})\ *\ \mathsf{dlist}\,\beta\,(\mathsf{l}, \mathsf{k}, \mathbf{nil}, \mathsf{m})\}$

if $\mathsf{j} = \mathbf{nil}$ **then**

$\{\mathsf{dlist}\,\beta\,(\mathsf{l}, \mathsf{k}, \mathbf{nil}, \mathsf{m}) \wedge \mathsf{i} = \mathsf{k} \wedge \mathbf{nil} = \mathsf{j} \wedge \alpha = \epsilon\}$

$\mathsf{i} := \mathsf{l}$

$\{\mathsf{dlist}\,\beta\,(\mathsf{l}, \mathsf{k}, \mathbf{nil}, \mathsf{m}) \wedge \mathsf{i} = \mathsf{l} \wedge \mathbf{nil} = \mathsf{j} \wedge \alpha = \epsilon\}$

else

$\{\exists \alpha', \mathsf{a}, \mathsf{n}.\ (\mathsf{dlist}\,\alpha'\,(\mathsf{i}, \mathbf{nil}, \mathsf{j}, \mathsf{n})\ *\ \mathsf{j} \mapsto \mathsf{a}, \mathsf{k}, \mathsf{n}$

$*\ \mathsf{dlist}\,\beta\,(\mathsf{l}, \mathsf{k}, \mathbf{nil}, \mathsf{m})) \wedge \alpha = \alpha' \cdot \mathsf{a}\}$

$[\mathsf{j}+1] := \mathsf{l}\ ;$

$\{\exists \alpha', \mathsf{a}, \mathsf{n}.\ (\mathsf{dlist}\,\alpha'\,(\mathsf{i}, \mathbf{nil}, \mathsf{j}, \mathsf{n})\ *\ \mathsf{j} \mapsto \mathsf{a}, \mathsf{l}, \mathsf{n}$

$*\ \mathsf{dlist}\,\beta\,(\mathsf{l}, \mathsf{k}, \mathbf{nil}, \mathsf{m})) \wedge \alpha = \alpha' \cdot \mathsf{a}\}$

$\{\mathsf{dlist}\,\alpha\,(\mathsf{i}, \mathbf{nil}, \mathsf{l}, \mathsf{j})\ *\ \mathsf{dlist}\,\beta\,(\mathsf{l}, \mathsf{k}, \mathbf{nil}, \mathsf{m})\}$

$\vdots$

Deleting from a Doubly-Linked List (continued)

$$
\begin{array}{l}
\quad \vdots \\
\{\mathsf{dlist}\ \alpha\ (\mathsf{i}, \mathbf{nil}, \mathsf{l}, \mathsf{j})\ *\ \mathsf{dlist}\ \beta\ (\mathsf{l}, \mathsf{k}, \mathbf{nil}, \mathsf{m})\} \\
\mathbf{if}\ \mathsf{l} = \mathbf{nil}\ \mathbf{then} \\
\quad \{\mathsf{dlist}\ \alpha\ (\mathsf{i}, \mathbf{nil}, \mathsf{l}, \mathsf{j}) \wedge \mathsf{l} = \mathbf{nil} \wedge \mathsf{k} = \mathsf{m} \wedge \beta = \epsilon\} \\
\quad \mathsf{m} := \mathsf{j} \\
\quad \{\mathsf{dlist}\ \alpha\ (\mathsf{i}, \mathbf{nil}, \mathsf{l}, \mathsf{j}) \wedge \mathsf{l} = \mathbf{nil} \wedge \mathsf{j} = \mathsf{m} \wedge \beta = \epsilon\} \\
\mathbf{else} \\
\quad \{\exists \mathsf{a}, \beta', \mathsf{n}.\ (\mathsf{dlist}\ \alpha\ (\mathsf{i}, \mathbf{nil}, \mathsf{l}, \mathsf{j})\ *\ \mathsf{l} \mapsto \mathsf{a}, \mathsf{n}, \mathsf{k} \\
\qquad *\ \mathsf{dlist}\ \beta'\ (\mathsf{n}, \mathsf{l}, \mathbf{nil}, \mathsf{m})) \wedge \beta = \mathsf{a}{\cdot}\beta'\} \\
\quad [\mathsf{l} + 2] := \mathsf{j} \\
\quad \{\exists \mathsf{a}, \beta', \mathsf{n}.\ (\mathsf{dlist}\ \alpha\ (\mathsf{i}, \mathbf{nil}, \mathsf{l}, \mathsf{j})\ *\ \mathsf{l} \mapsto \mathsf{a}, \mathsf{n}, \mathsf{j} \\
\qquad *\ \mathsf{dlist}\ \beta'\ (\mathsf{n}, \mathsf{l}, \mathbf{nil}, \mathsf{m})) \wedge \beta = \mathsf{a}{\cdot}\beta'\} \\
\{\mathsf{dlist}\ \alpha\ (\mathsf{i}, \mathbf{nil}, \mathsf{l}, \mathsf{j})\ *\ \mathsf{dlist}\ \beta\ (\mathsf{l}, \mathsf{j}, \mathbf{nil}, \mathsf{m})\} \\
\{\mathsf{dlist}\ \alpha{\cdot}\beta\ (\mathsf{i}, \mathbf{nil}, \mathbf{nil}, \mathsf{m})\}
\end{array}
$$

Trees (Binary)

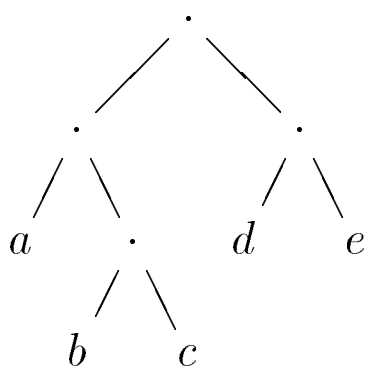
$\tau \in$ Trees iff

$$\tau \in \text{Atoms}$$

$$\text{or } \tau = (\tau_1, \tau_2) \text{ where } \tau_1, \tau_2 \in \text{Trees}$$

e.g.

$((a,(b,c)),(d,e)) =$



Tree Segments

$\tau \in \text{Trees}(n)$ iff

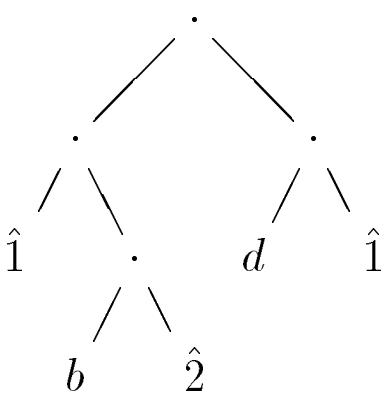
$$\tau = \hat{k} \text{ where } 1 \leq k \leq n$$

$$\tau \in \text{Atoms}$$

$$\text{or } \tau = (\tau_1, \tau_2) \text{ where } \tau_1, \tau_2 \in \text{Trees}(n)$$

e.g.

$((\hat{1},(b,\hat{2})),(d,\hat{1})) =$



Tree Representation (without sharing)

For $\tau \in$ Trees(n), we define the assertion

$$\mathsf{tree}\ \tau\ (i : j_1, \ldots\ j_n)$$

by structural induction:

$$\mathsf{tree}\ \hat{k}\ (i : j_1, \ldots\ j_n)\ \text{iff}\ \mathbf{emptyheap} \wedge i = j_k$$

$$\mathsf{tree}\ a\ (i : j_1, \ldots\ j_n)\ \text{iff}\ \mathbf{emptyheap} \wedge i = a$$

$$\mathsf{tree}\ (\tau_1, \tau_2)\ (i : j_1, \ldots\ j_n)\ \text{iff}\ \exists i_1, i_2.\ i \mapsto i_1, i_2$$
$$* \ \mathsf{tree}\ \tau_1\ (i_1 : j_1, \ldots\ j_n)\ * \ \mathsf{tree}\ \tau_2\ (i_2 : j_1, \ldots\ j_n)$$

Tree Representation (with sharing)

For $\tau \in$ Trees(n), we define

$$\mathsf{stree}\ \tau\ (i : j_1, \ldots\ j_n)$$

by:

$$\mathsf{stree}\ \hat{k}\ (i : j_1, \ldots\ j_n)\ \text{iff}\ i = j_k$$

$$\mathsf{stree}\ a\ (i : j_1, \ldots\ j_n)\ \text{iff}\ i = a$$

$$\mathsf{stree}\ (\tau_1, \tau_2)\ (i : j_1, \ldots\ j_n)\ \text{iff}\ \exists i_1, i_2.\ i \mapsto i_1, i_2$$
$$* \ \Big(\mathsf{stree}\ \tau_1\ (i_1 : j_1, \ldots\ j_n) \wedge \mathsf{stree}\ \tau_2\ (i_2 : j_1, \ldots\ j_n)\Big)$$

A Problem

Suppose we wish to prove that a recursively defined procedure copytree satisfies

$$\{\mathsf{stree}\ \tau(\mathsf{i}:)\}$$
$$\mathsf{copytree}(\mathsf{i},\mathsf{j})$$
$$\{\mathsf{stree}\ \tau(\mathsf{i}:)\ *\ \mathsf{tree}\ \tau(\mathsf{j}:)\}$$

Then, we must use this specification as a hypothesis in proving that the recursive calls in the procedure body satisfy:

$\{\mathsf{stree}\ \tau_1(\mathsf{i}_1:) \wedge \mathsf{stree}\ \tau_2(\mathsf{i}_2:)\}$

$\mathsf{copytree}(\mathsf{i}_1,\mathsf{j}_1)$

$\{(\mathsf{stree}\ \tau_1(\mathsf{i}_1:) \wedge \mathsf{stree}\ \tau_2(\mathsf{i}_2:))\ *\ \mathsf{tree}\ \tau_1(\mathsf{j}_1:)\}$

$\mathsf{copytree}(\mathsf{i}_2,\mathsf{j}_2)$

$\{(\mathsf{stree}\ \tau_1(\mathsf{i}_1:) \wedge \mathsf{stree}\ \tau_2(\mathsf{i}_2:))\ *\ \mathsf{tree}\ \tau_1(\mathsf{j}_1:)\ *\ \mathsf{tree}\ \tau_2(\mathsf{j}_2:)\}$

But the hypothesis is not strong enough to imply this.

Possible solutions:

- Introduce passivity.
- Introduce auxiliary variables denoting assertions:

$$\{\mathsf{stree}\ \tau(\mathsf{i}:) \wedge \mathsf{p}\}$$
$$\mathsf{copytree}(\mathsf{i},\mathsf{j})$$
$$\{(\mathsf{stree}\ \tau(\mathsf{i}:) \wedge \mathsf{p})\ *\ \mathsf{tree}\ \tau(\mathsf{j}:)\}$$

- Introduce auxiliary variables denoting heaps: $\mathbf{this}(\mathsf{h})$.

Iterated Independent Conjunction

$$m > n \Rightarrow \left(\bigodot_{i=m}^{n} p(i) \Leftrightarrow \mathbf{emptyheap}\right)$$
$$m = n \Rightarrow \left(\bigodot_{i=m}^{n} p(i) \Leftrightarrow p(m)\right)$$
$$k \leq m \leq n+1 \Rightarrow \left(\bigodot_{i=k}^{n} p(i) \Leftrightarrow \left(\bigodot_{i=k}^{m-1} p(i) \ * \ \bigodot_{i=m}^{n} p(i)\right)\right)$$

An Inference Rule for Array Allocation

$$\{p\}$$
$$v := \mathbf{allocate}\ e$$
$$\{p \ * \ \bigodot_{i=v}^{v+e-1} i \mapsto -\}$$

where v does not occur free in p or e.

Future Directions

- Directed graphs
- Dynamic logic and refinement calculi
- Assertions versus types
- Garbage collection

$$
\begin{array}{l}
\{\mathbf{emptyheap}\} \\
\mathsf{x} := \mathbf{cons}_2(1,2)\ ; \\
\{\mathsf{x} \rightarrow 1,2\} \\
\mathsf{x} := 3 \\
\{(\exists \mathsf{x}.\ \mathsf{x} \rightarrow 1,2) \wedge \mathsf{x} = 3\}
\end{array}
$$

- Concurrency

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